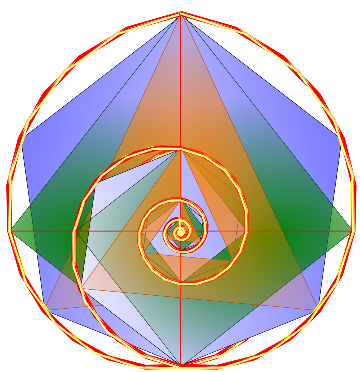
INTIMATIONS OF COMMONALITY IN PLANETARY SYSTEMS



2023

EXCURSUS

HISTORICAL ISSUES & PROBLEMS
(OPTIONAL)

I. The Babylonian Ecliptic Reference Frame and the Zodiac

Apart from the relatively recent *Astronomical Diaries and Related Texts* by Abraham J. Sachs and Hermann Hunger (1988) this is still a neglected area. Yet there is literally nothing in the historical record which comes remotely close to details in mathematical cuneiform texts from the Old Babylonian Period (1900 BCE-1650 BCE), the Ephemerides, the various "Procedure" texts of the Seleucid Era (310 BCE - 75 CE) and the Babylonian Astronomical Diaries from 625 BCE - 65 BCE. Included among the latter are the two frames of reference against which the major astronomical phenomena take place; 1. The local horizon, and 2. A fixed celestial reference frame provided by +33 bright stars distributed around the ecliptic, *i.e.*, within a few degrees of the paths of the visible planets.

The *Diaries* represent a prime resource while also including background information on the subject plus three sets of longitudes (L) and inclinations (i) of the Babylonian ecliptic reference stars for 600 BCE, 300 BCE and the Year 0 CE. For more than enough data, in fact, to display the zodiacal reference frame and the ecliptic "normal" stars with their various names and signs against those published in Neugebauer's *Astronomical Cuneiform Texts* (1955) to overcome the difficulties caused by the latter's overly condensed and far from helpful presentations of both the texts and their contents. Thus, starting with Figure 2a below, the dual names and signs for the Zodiac used in this major reference followed by a condensed presentation of Babylonian planetary parameters and periods relations on the next page.

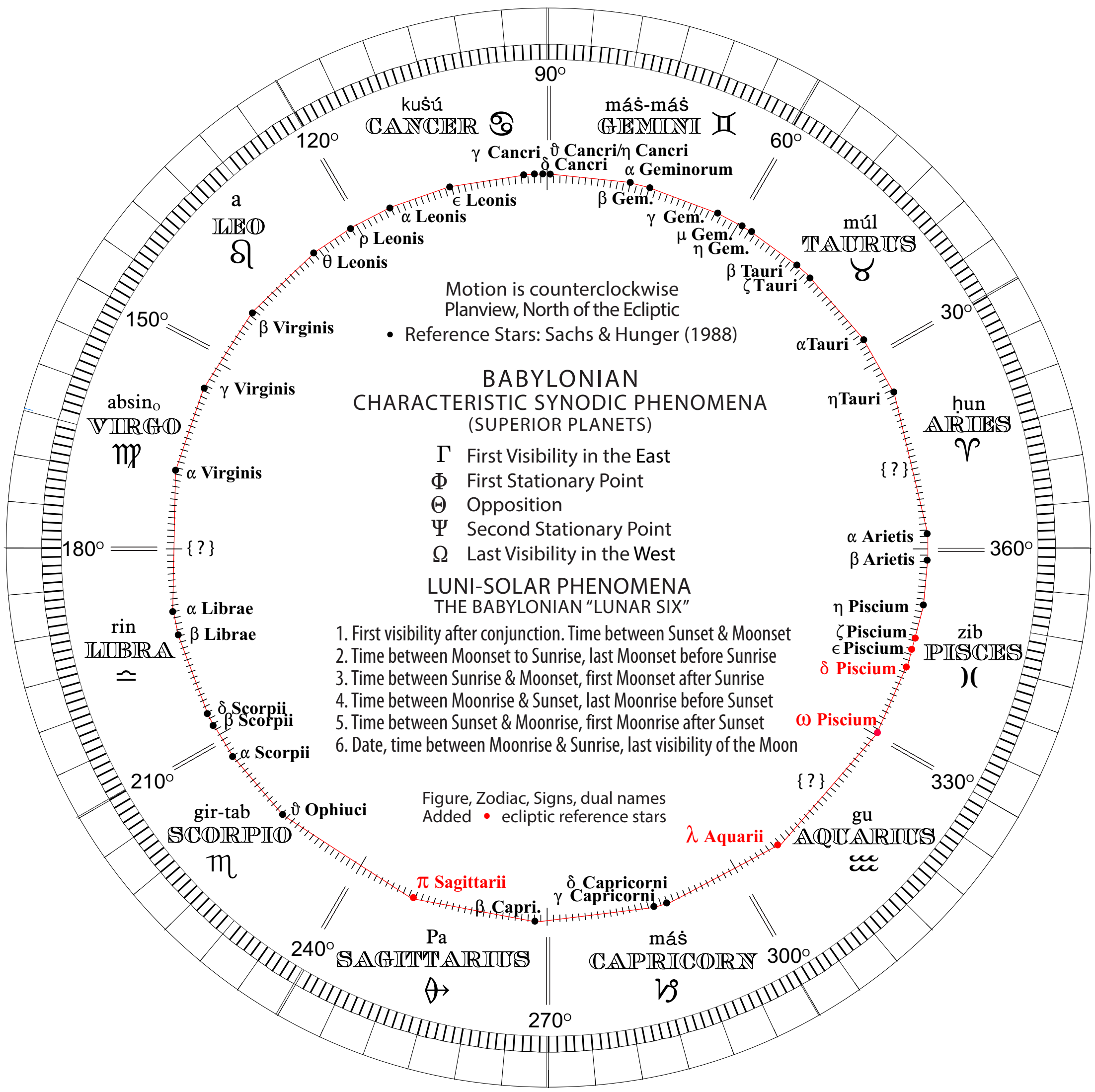


Fig. 2a. Babylonian ecliptic "Normal" Stars; the Zodiac plus standard signs and dual names

♈ Aries	(ḫun)	0°	♌ Leo	(a)	120°	♐ Sagittarius	(Pa)	240°
♉ Taurus	(múl)	30°	♍ Virgo	(absin₀)	150°	♑ Capricorn	(máš)	270°
♊ Gemini	(máš-máš)	60°	♎ Libra	(rin)	180°	♒ Aquarius	(gu)	300°
♋ Cancer	(kušú)	90°	♏ Scorpio	(gir-tab)	210°	♓ Pisces	(zib)	330°

I. BABYLONIAN PLANETARY PARAMETERS AND PERIOD RELATIONS

The five planets known in Antiquity are shown in planview north of the ecliptic in heliocentric order with Earth (following Mercury and Venus) third planet from the Sun. Also supplied are modern estimates for the mean periods of revolution and calculated mean synodic periods/lap-cycles from relations (2s) and (2i) given on page 2.

A. Mean Periods of revolution (*T*) [6]

N[1]	PLANET Sidereal	Modern T revolutions[2]	Babylonian revolutions	Babylonian Ratio: N/Z	Diff. (T) % Diff.
3	Saturn	29.423519	29.444444	(265/9)	-2.093%
4	Jupiter	11.856520	11.861111	(427/36)	-0.459%
5	[M-J Gap]	-	-	-	-
6	Mars	1.8807111	1.8807947	(284/151)	0.0084%
-	[Earth]	1	1	[1]	-
7	Venus	0.6151857	0.6151790 [2]	(1151/1871)	0.0004%
8	Mercury	0.2408444	0.2408377 [2]	(46/191)	0.0007%

[1] N Planets counted inwards from Neptune. [2] Revolutions added.

B. Mean Synodic periods/Lap cycles (*S*) [7]

N	PLANET Synodic	Modern S Cycle (years)	Babylonian Cycle (years)	Babylonian Ratio: N/II	Diff. (S) % Diff.
3	Saturn	1.0351821	1.0351563	(265/256)	0.003%
4	Jupiter	1.0921105	1.0920716	(427/391)	0.004%
5	[M-J Gap]	-	-	-	-
6	Mars	2.1354462	2.1353384	(284/133)	0.011%
-	[Earth]	-	-	-	-
7	Venus	1.5986349	1.5986111	(1151/720)	0.002%
8	Mercury	0.3172531	0.3172414	(46/145)	0.001%

Table 4. Modern and Babylonian mean periods *T, S* in years

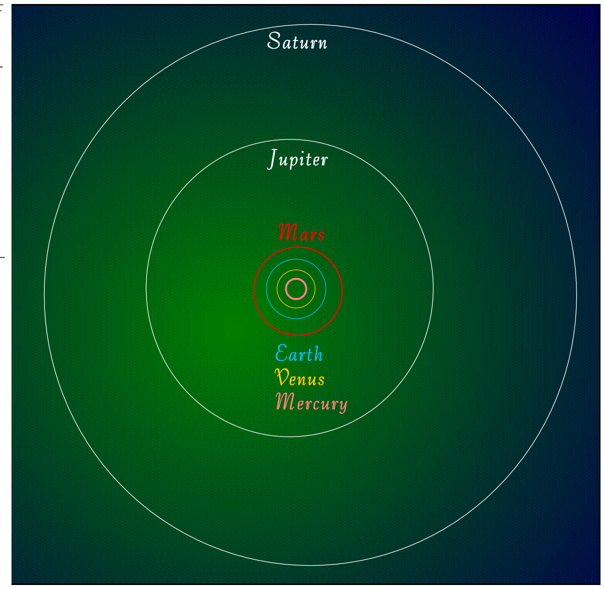


Fig.3. The Orbits of the five planets known in Antiquity

Synodic relations (1), (2) and the modern equivalents all have roles to play in Table 4, but relation (1) in full has a further application arising from the inclusion of the mean synodic month in Babylonian planetary theory beyond mere calendaric considerations. More on this later.

Babylonian mean periods, on the other hand, are from data-sets which provide insights into their determination, initial lack of revolutions for the two inferior planets included. As they now stand and as presented in ACT (1955) by Neugebauer, Babylonian mean periods were, however, seemingly "simple" integer period relations in a dual form:[28]

Superior planets: N years : II Synodic arcs/periods : Z Rotations;	N Years = II Synodic arcs + Z Rotations
Inferior planets: N years : II Synodic arcs/periods. (only)	II Synodic arcs in N Years (no rotations)

whereas, in contrast to Neugebauer's treatment, from a fictive, heliocentric viewpoint and *revolutions* substituted for "*rotations"* plus the mean *periods of revolution (Z) added* for Venus and Mercury, a more complete and balanced arrangement is:

THE SUPERIOR PLANETS: SATURN, JUPITER AND MARS
SATURN: 265 years - 256 Synodic arcs = 9 Revolutions
JUPITER: 427 years - 391 Synodic arcs = 36 Revolutions
MARS: 284 years -133 Synodic arcs = 151 Revolutions

THE INFERIOR PLANETS: VENUS AND MERCURY
VENUS: 1151 years + 720 Synodic arcs = *1871 Revolutions (added)*
MERCURY: 46 years + 145 Synodic arcs = *191 Revolutions (added)*
Also (shown later) four more period relations for Mercury are attested.

which yields by integer division results which are comparable to the modern values shown in Table 4.

II. Babylonian "characteristic" synodic phenomena

From a modern viewpoint it is easiest to start with Jupiter at Opposition to introduce the "characteristic" synodic phenomena since of the five Babylonian synodic events it is only at Opposition that the *location* of Jupiter is at its *true position*. Whereas, for three of the other four phenomena the locations are merely *apparent, w*hile for the fourth (from the Last Visibility in the West to the First Visibility in the East) Jupiter is moving on the opposite side of the Solar System and cannot be observed at all throughout the interval. For this reason the following figures include the annual *sidereal* motion for Jupiter with Figure 4a initially assigned an integer value of 12 years for the period of revolution (*T*) before including the Babylonian characteristic phenomena as an integral set in Figure 4b.

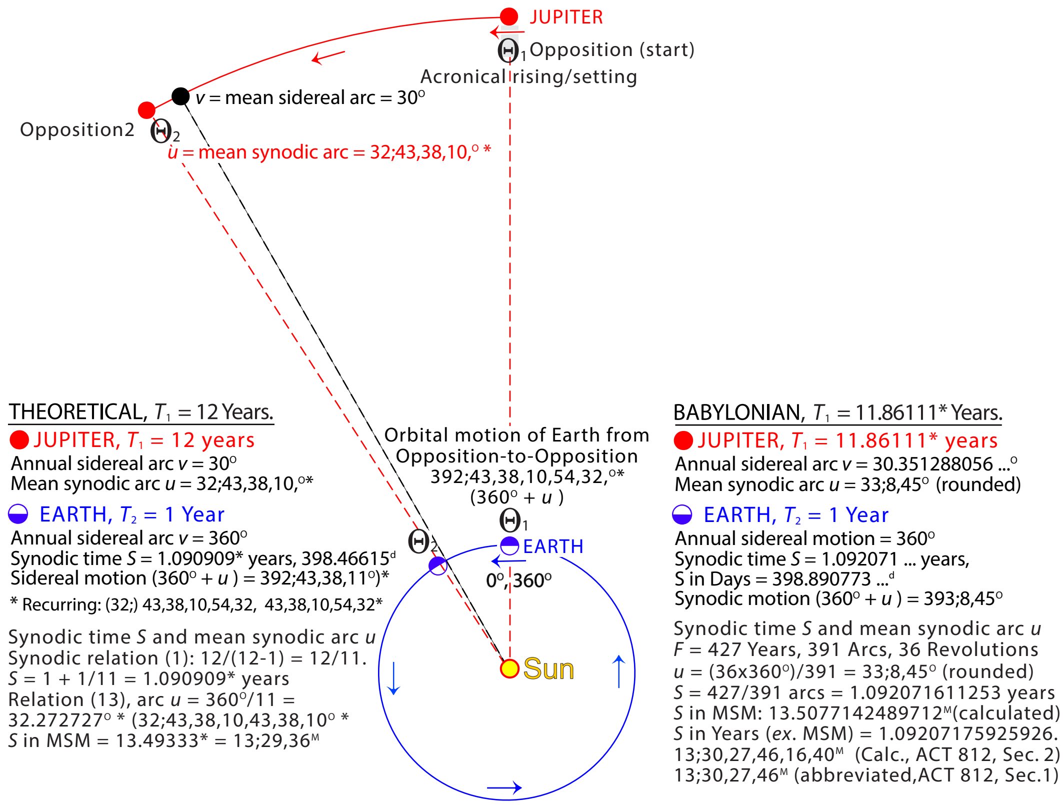


Fig. 4a. The mean synodic arc u for Jupiter, theoretical T_1 12 years. Babylonian T_1 as used 11.86111* years.

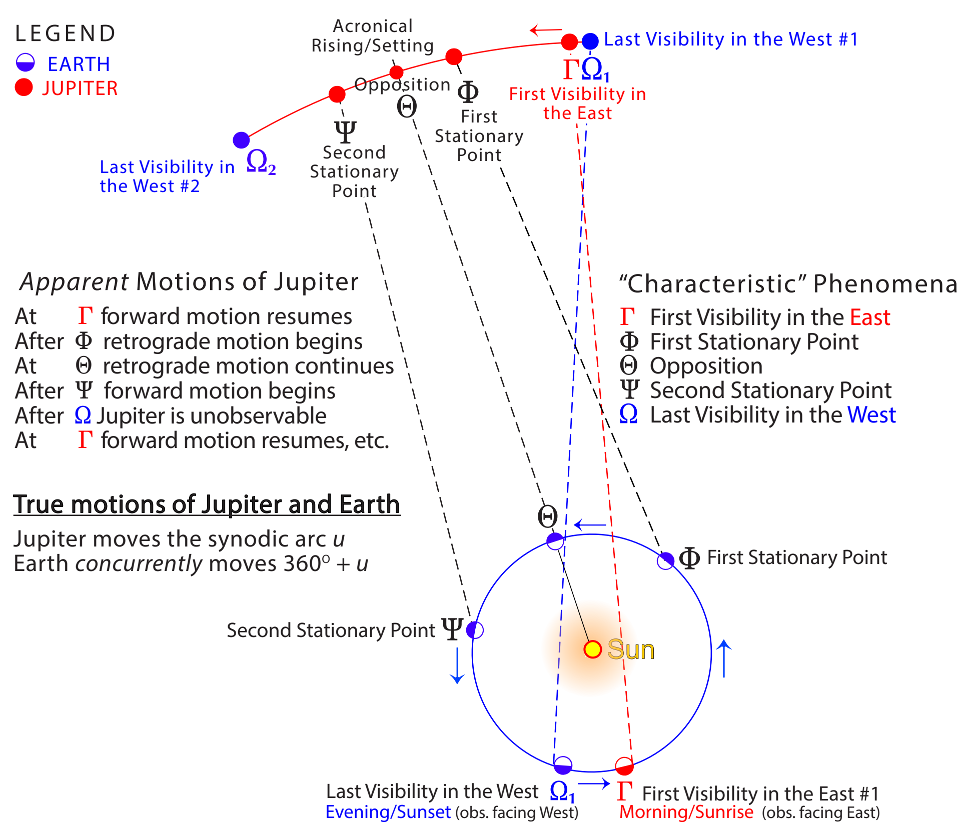


Fig. 4b. Jupiter: Characteristic Phenomena and concurrent heliocentric motion of Earth.

Shown in planview north of the ecliptic in Figure 6, the Babylonian reference frames enclose the "characteristic" synodic phenomena for Jupiter on the 34° synodic arc accompanied by the direct heliocentric motion of Earth. As perhaps stated in Inset A and Note 1: [29]

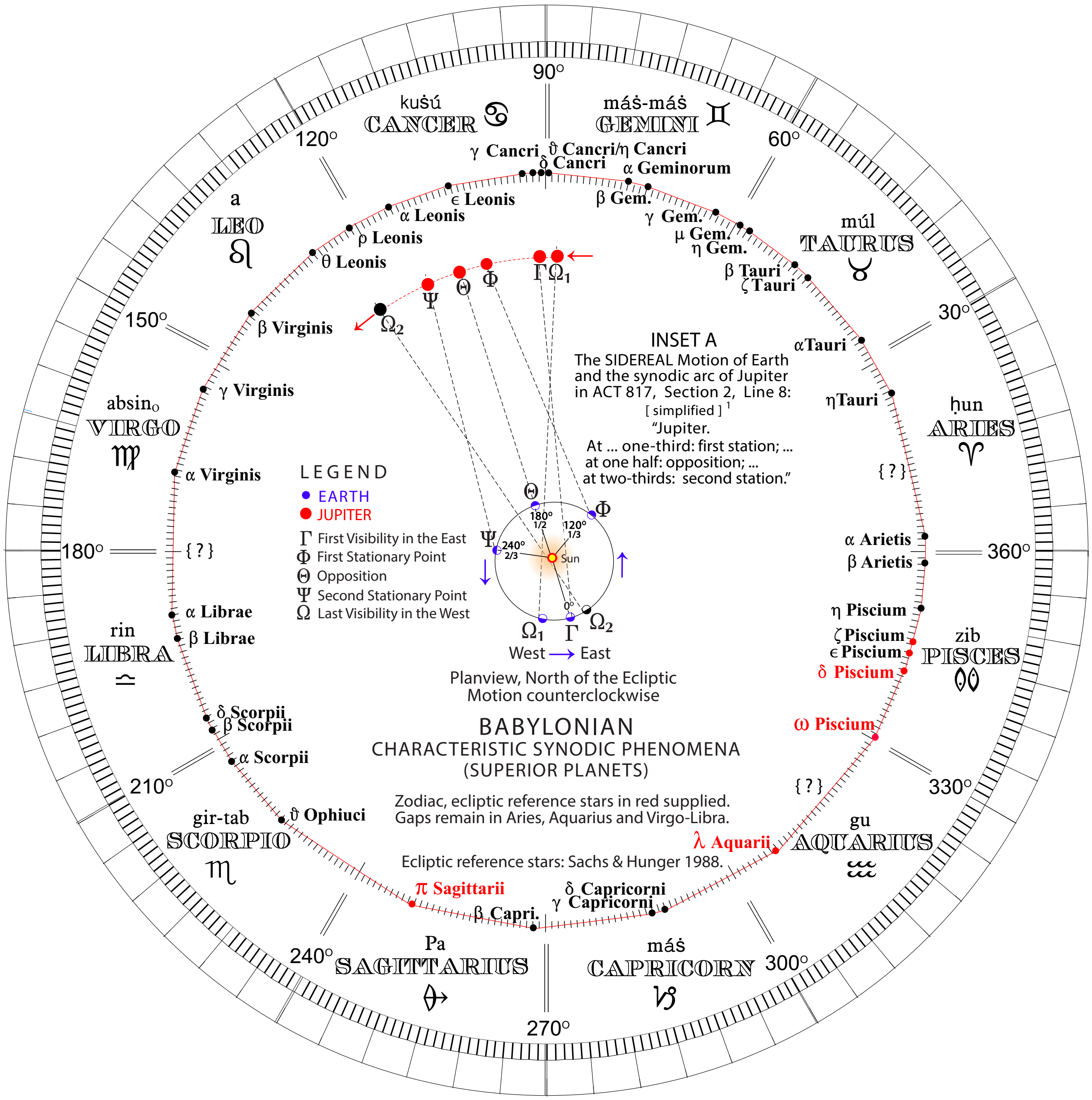


Fig 5. The Zodiac, Babylonian Ecliptic Reference Stars, Synodic Phenomena, Heliocentric motion of Earth.

1. ACT 817, Section 2, Line 8 in full: "Dr. Sachs suggested the following interpretation of line 8: ' Jupiter. At one-third of its (stretch of) visibility: first station; at one-half of its (stretch of visibility: [Opposition; at two-thirds: second station]'. This is indeed in good agreement with the general schemes for subdividing the synodic arc of Jupiter." Otto Neugebauer, ACT (1955: 430).

But It is *EARTH* which is moving approximately "one-third," then "one half," followed by "two-thirds" over the full synodic arc of 360°+*u*. Plus the period for *Earth* to move from the Last visibility of Jupiter in the West (Ω_1) to the First visibility of Jupiter in the East (Γ) from ACT 813, Section 11 is 27 days (in Cancer ♋) and 32 days (in Sagittarius ♐).[30] Thus a mean value of 29.5 days for the *varying* intervals of invisibility *versus* the mean synodic month (29. 5306 days) with regular variations arising from the elliptical orbits of Moon, Earth and Jupiter. Plus considerable problems from parallax to further complicate the precise location of Jupiter on the day of the First visibility in the East.

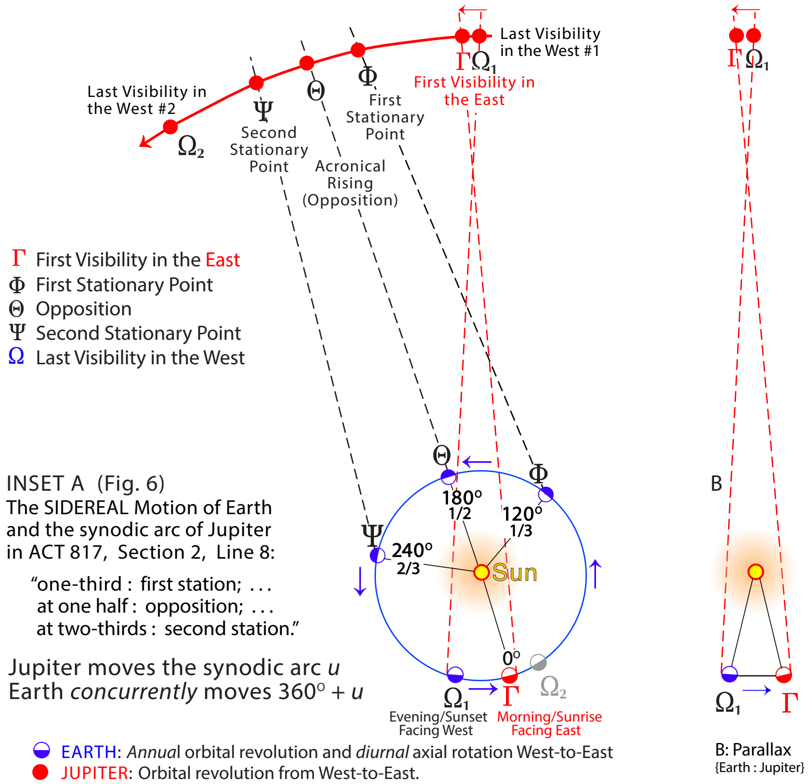


Fig. 6. Heliocentric motions of Jupiter and Earth from line 8, Section 2 of ACT 817.

III. The determination of the Babylonian Integer Period Relations

Shown in plan view north of the ecliptic it has long been recognised that the relatively small eccentricities of the Solar System planets cause the orbits to resemble slightly off-set circles as opposed to elliptical orbits *per se*.

What might all this have to do with Babylonian planetary theory? A great deal, as it turns out, because - as in the case of the mean periods - modern eccentricities also have an *a priori* element to them. In other words, the mean distances and the eccentricities are *given* with the apsidal distances calculated from this base with respect to an *already assigned line of apsides*. Thus the locations of the shortest (and fastest moving) radius vectors/distances and the longest (also slowest moving) radius vectors for each planet. Whereas the Babylonians, having determined the mean periods and associated mean velocities from a lengthy series of observations, next seem to have derived the values for not only for the extremal synodic arcs, but also precise locations for the same. In short, lines of apsides. But first, how were the integer period relations which provided the Babylonian periods (*T*, *S*) in Table 4a obtained?

Fortunately, the origin of the integer relations is generally well known, even if the impressive results are not. Or perhaps better stated, still largely driven into obscurity by partial and less than praiseworthy modern analyses that fail to mention the accuracy so obtained. Nor do repeated references to "rotations" help here either, or an inability to consider the significance of the difference in format for the final relations of the Superior and Inferior planets.

Babylonian "long" period relations

The Final period relation for Jupiter: In 427 years: 391 synodic arcs and 36 Revolutions

The method for obtaining the mean and varying velocities and the associated times is explained in great detail in the procedure texts from the Seleucid Era, with those for Jupiter among the most informative, *e.g.*, as in ACT 813, a Babylonian "procedure" text for the determination of the *mean* synodic arc which in Sections 1 and 21 includes the simple integer period relation for Jupiter of 427 years as the end product: [31]

> [in 6,]42 days from (one) appearance to the (next) appearance (Jupiter) moves 30 degrees.' This is correct for the slow arc of System A' for Jupiter. For lines 4 and 5 Sachs suggests: "Compute(?) for the whole zodiac (or for each sign) the appearance according to the day and the velocity.' This is followed in lines 5 and 6 by the rules for the periods: 'in 12 years you add 4;10, in 1,11 (71 years) you subtract 5, in 7,7 (427 years) the longitude (returns) to its (original) longitude' . . . ' In the Almagest the correction for 71 years is - 4;50° (IX3, Heiberg p. 215; Manitius II p. 100.

The translation and reference to the *Almagest* are by Neugebauer; decimal periods have been added for clarity and "revolutions" have also been substituted for "rotations" uniformly promoted by the latter. In passing it is safe to say that the latter replacement(s) largely correct Neugebauer's generally inadequate terminology.

Origins and Implications

It is apparent from this procedure that for Jupiter a pair of integer periods were selected (named here *T*1 and *T*2) with corresponding near-integer numbers of synodic periods, near-integer numbers of revolutions with *small and convenient* corrections *in longitude of opposite sign.* From these periods and corrections the final Babylonian "long" period integer relation of 427 years is readily obtained by addition. As in fact suggested by the writer in a *Letter to the Editor* of *ISIS* published in 1977 [32] in an earlier attempt to close Babylonian orbits about the common center.

T	Periods (Years)	Progress T1 - T7	Synodic Synodic arcs	Sidereal Revolutions	Corrections (Longitude) Slow	Fast
T1	**12 years**	(12)	11	1 x 360°	+4;10°	+5;00°
T2	**71 years**	(6*12)	65	6 x 360°	-5;00°	+6;00°
T3	83 years	(12 + 71)	76	7 x 360°	- 0;50°	-1;00°
T4	95 years	(12 + 83)	87	8 x 360°	+3;20°	+4;00°
T5	166 years	(71 + 95)	152	14 x 360°	-1;40°	-2;00°
T6	261 years	(166 + 95)	239	22 x 360°	+1;40°	+2;00°
F7	**427 years**	(166 + 261)	**391**	**36** x 360°	**0;00°**	**0;00°**

Table 5. Babylonian Jupiter corrections and the final 427-year period ***F***.

Although Neugebauer was aware of the corrections and almost all of the periods which lead to **F7**, the final integer period relation, he did not pursue the matter further, although (on inspection) closed orbits are obviously involved since each correction is also *heliocentric* in nature. For example, by the same process used for the *synodic arc*, the *annual sidereal arc* for Jupiter is obtained from the division of the total sidereal motion (as before 36•360° = 12,960°) by 427, the number of years for ***F***, the final period. The *mean sidereal motion* for Jupiter (30.351288056° *per annum*) applied to the *T*1 and *T*2 periods gives the excess over 12 years (*T*1) and the deficiency after 71 years (*T*2) *modulo* 360° which results in values of 4.2152457° and -5.058548° (4;12,55,38, and -5;3,30,46,) thus rounded to the nearest sixth of a degree (0;10°) for both, values of 4;10° and -5;00° as used by the Babylonians.

The 427-year Final period for Jupiter

Clearly, from the above the initial stage of the Babylonian inquiry appears to have been the derivation of an integer number of years to which there corresponded an integer number of mean revolutions and a corresponding integer number of mean synodic periods. With this primary goal for Jupiter achieved, the next step following in Section 21 of ACT 813 is the determination of the mean synodic arc (*u*) as follows: [33]

> [7,7 years (correspond to) 6,31 appearances (Γ),] 36 rotations, 3,36,0° motion, 33;8,[4]5 (is the) mean value of the longitudes.

which in plain English and decimal notation is clearly the unambiguous statement that:

> 427 years correspond to 391 synodic periods, 36 *revolutions*, 12,960° *sidereal* motion, 33;8,45 (is the) mean value of the synodic arc. (added: with 33;8,44,*48,29*,°. . . . rounded to 33;8,45°).

Thus, unequivocally, 36 revolutions of mean circular motion with a subdivided arc subtended from the center with 427 years also the 427 revolutions of Earth. And as seen next, the same argument is applicable to the division of the total sidereal motion of Saturn by the corresponding number of mean synodic periods to derive the mean synodic arc for this planet in the same manner. Except here the result is exact and convenient in both base-60 *and* base-10.

The 265-year Final period relation for Saturn

The identical procedure is given for Saturn in abbreviated form with the total sidereal motion supplied in Section 4 of ACT 802, a Saturn procedure text. Which in decimals with *revolutions* substituted for "rotations," is: [34]

> Concerning Saturn. 265 years corresponds to 256 synodic periods, 9 sidereal revolutions and 3,240° total motion.

leading to the mean synodic arc for Saturn *u* (12;39,22,30°) from the total sidereal motion (9•360° = 3,240°) divided by the number of mean synodic periods (256). Section 5 supplies the constants for the extremal synodic periods, and Section 6 the corresponding extremal synodic arcs. The two intermediate periods *T*1 and *T*2 with corrections of opposite sign for Saturn are 29 years and 59 years which lead in turn to the 265-year integer relation ***F*** of Saturn.

The 284-year Final period relation for Mars

For Mars $T1$ = 47 years, $T2$ = 79 years with the Final integer period F_n = 284 years, to which correspond 151 x (360°) periods of revolution and 133 mean synodic arcs/mean synodic periods. The total motion is again divided by the number of synodic events (151•360°= 54,360°) divided by 133 = 408;43,18,29,46, . . . abbreviated to 408;43,18,30° and ultimately u = 48;43,18,30°. This differs markedly from Jupiter and Saturn where Earth moves (360°+u) in the time that it takes for these two outer planets to move the mean synodic arc u alone. Now, to lap Mars, Earth must move (2•360°+u) during which time Mars itself moves (1•360°+u). This departure from the format for both Saturn and Jupiter is perhaps one reason for earlier uncertainty concerning any Babylonian System B velocity function for Mars despite hints in Mars procedure text ACT 811a.[35]

IV. The Inferior planets Venus and Mercury.

Venus

In addition to the well-known but barely recognized *Fibonacci* pairing of 5 synodic periods/arcs in 8 years, a longer Final period of 720 synodic periods/arcs in 1151 years is also attested. The number of revolutions for this planet are again absent, with the final integer relation merely the numbers of years and corresponding numbers of synodic arcs/periods: [36]

Venus: 8 years corresponds to 5 synodic arcs/periods
Venus: 1151 years corresponds to 720 synodic arcs/periods

with the ratio of the two parameters providing the mean synodic period(s) in years: 8/5 = 1.6, and more accurate 1151/720 = 1.5986111*years. Expressed in days the latter mean synodic period for Venus is 583.9097^d versus the modern estimate of 583.9214^d.

Mercury

Five Babylonian period relations for Mercury are attested with 145 synodic periods in 46 year the best known.[37] As in the case for Venus the corresponding number of revolutions is not stated, an *apparent* deficiency maintained by Claudius Ptolemy in the *Almagest* and a mistake also retained by almost all subsequent commentators.

The resulting mean synodic period for Mercury is quite accurate, *i.e*., 46/145 = 0.317241 years and 115.8758 days versus the modern estimate of 115.8775 days. The same degree of accuracy is also provided by the four remaining integer period relations. The latter set of limited Babylonian period relations for Mercury replete with Greek-letter assignments provided by Neugebauer are included below with the missing numbers of sidereal revolutions and the missing revolutions $(Z = T)$ supplied.[38] The period of revolution (T) for the 46-year, 145 synodic period relation (and corresponding 191 revolutions) is 0.24083769 years ; the modern estimate for T is 0.24084445 years.

Σ In 388 years, 1223 disappearances as a morning star (and 1611 revolutions, T = 0.24084419 years)
Ξ In 480 years, 1513 appearances as an evening star (and 1993 revolutions, T = 0.24084295 years)
Ω In 217 years, 684 disappearances as an evening star(and 901 revolutions, T = 0.24084350 years)
Γ In 848 years, 2673 appearances as a morning star (and 3521 revolutions, T = 0.24084067 years)

Thus the 46-year period relation for Mercury does not even provide the best result, nor in their original Babylonian context do any of the *integer* period relations require a "correction," despite the later "improvements" by Claudius Ptolemy carried forward with the latter's defective geocentric planetary model to the time of Copernicus.

VI. The significance of the unstated periods of revolution for the Inferior planets

Although the final period (F) differs in format for the two Inferior planets from those applied for Mars, Jupiter and and Saturn, the lack of stipulated numbers of periods of revolution for Venus and Mercury is nonetheless merely a function of the heliocentric nature of Babylonian planetary theory. In other words, though the motion of Earth provides the frames of reference for both the times and the velocities, it is always the outer planet of the respective synodic pair that provides the synodic arc. For the superior planets it is Earth which laps Mars, Jupiter and Saturn during each synodic period, *i.e*., each lap-cycle:

SUPERIOR PLANETS

From the Number of Years, corresponding numbers of Synodic arcs/periods and numbers of Revolutions
Outer and Slower-moving SATURN, and JUPITER move the direct (synodic) arc (u).
Inner and Faster-moving EARTH concurrently moves (360° + u) to supply the time per lap-cycle.
Mars is the exception where Earth moves (2•360°+u) during which time Mars moves (1•360°+u).

1. EARTH *in all cases supplies the mean synodic time* per lap-cycle, *but not the mean synodic arc* (u).

INFERIOR PLANETS
From the relation: Number of Years and corresponding number of Synodic arcs/periods only
Outer and Slower-moving EARTH moves the direct (synodic) arc (*u*).
while inner and faster-moving VENUS and MERCURY *concurrently* move (n•360° + *u*) per lap-cycle.
1. EARTH *supplies the mean synodic time per lap-cycle*. 2. EARTH *also supplies the mean synodic arc (u).*

VENUS: In 1151 years, 720 synodic cycles (and unstated: 1871 revolutions)
Mean Synodic time: 1151/720 = 1.5986111* (1;35,55) years, or 583.909714 days.
Mean synodic arc (*u*): (1151•360°/720) = 575.5° and/or 215.5° from (1•360° + *u*).
during which time VENUS *concurrently moves* (2•360° + *u*).

MERCURY: In 46 years, 145 synodic cycles (and unstated: 191 revolutions)
Mean Synodic time: 46/145 = 0.3172413793 years or 115.8757885 days.
Mean synodic arc (*u*): (46•360°/145), *u* = 114.2068965°.
during which time Mercury *concurrently* moves (1•360° + *u*).

Thus from the heliocentric viewpoint, for the inner planets Earth is now supplying the synodic arc as an outer planet, but still providing the synodic time as before. Thus a variant of the known relation for the Superior planets where the number of revolutions are components of the integer relation itself. Therefore, remaining with the terminology and the symbols used by Neugebauer (albeit with *revolutions* substituted for *rotations*), the primary Superior planet relation necessarily includes the following re-arrangement of terms:

SUPERIOR PLANETS. Variant of the primary period relation for Saturn, Jupiter and Mars:
Number of Sidereal Revolutions (Z) = Number of Years (N) - Number of Synodic periods (II)

e.g., returning to Jupiter, 36 revolutions = (427 years-391 synodic periods) with the mean synodic arc (*u*) = 33;8,45° derived and rounded as given in ACT 813, Section 21. Although not needed for practical reasons, should the actual periods of revolution for Venus or Mercury be required they can be readily obtained from the following variant:

INFERIOR PLANETS. Variant of the primary period relation
Number of Sidereal Revolutions (Z) = Number of Synodic periods (II) + Number of Years (N)

Thus for Venus the Fibonacci series itself, *i.e*.,

5 Synodic periods + **8** Years = **13** sidereal revolutions (5 + 8 + 13, etc.),

with the mean period of revolution (Z) for Venus from: 8/13 = 0.615385 years, versus:
1. The modern estimate = 0.615186 years
2. Venus from 1151/1871 = 0.615179 years
3. The reciprocal of *Phi* = 0.618034 years

VII. Babylonian astronomy: a modern enigma

Lastly, although the astronomical assignments for the Old Babylonian Period in Text-Fig 1are at odds with prevailing chronology and understanding they are not made lightly or without supporting research and textual support.

Nevertheless, as far as Babylonian astronomy is concerned, primary modern works by Otto Neugebauer (1955,[39] 1975[40]), Bartel van der Waerden (1974)[41] and Noel Swerdlow (1998)[42] accept the odd premise - despite procedures and details in the astronomical cuneiform texts - that the Babylonians possessed no fictive planetary model at all.

In fact, by the time of his final *opus, A History of Ancient Mathematical Astronomy* (1975) Neugebauer states with ill-founded certitude that the understanding of Babylonian astronomy :[43]

> is a rather self-contained problem of great technical complexity that will occupy us in the following chapters. *But it will reveal to us the working of a theoretical astronomy* which operates *without any model of a spherical universe, without circular motions and all the other concepts which seemed "a priori " necessary for the investigation of celestial phenomena. The influence of the available mathematical tools on the development of a scientific theory is perhaps far greater than we customarily admit*.
> Otto Neugebauer, *A History of Mathematical Astronomy* (1975:347-348; italics supplied)

Unfortunately, this nihilistic, pre-conclusive viewpoint in consort with base-60 mathematics, unfamiliar procedures, hard to follow names and concepts has rendered the Babylonian material difficult to read in the first place and not worth reading in the second. Hence the unwarranted yet still prevailing ignorance on the matter.

Quite literally, Neugebauer "wrote the book on the subject," *i.e*., *Astronomical Cuneiform Texts* published in1955, but this last assessment is not only erroneous, it also leaves one wondering how Neugebauer managed to escape critical peer reviews, the opposing discussion of precession and Babylonian years by W. Hartner (1979)[26] excepted. Yet according to Neugebauer Babylonian astronomy exists without "any model" and "without circular motions and all other concepts . . . . necessary for the investigation of celestial phenomena," which is readily countered by the

Babylonian methods for the determination of fundamental period relations and mean synodic arcs provided in the various "procedure" texts of the Seleucid Era. Except that this material - replete with commentaries and notes - is in Neugebauer's *Astronomical Cuneiform Texts* (1955), and furthermore, key Babylonian procedures in this initial work either remain erroneous or incomplete. Whereas a further impediment to understanding - also introduced by the latter in his latest compendium - was a far from helpful caution, namely that: [44]

> In working with sexagesimally written texts it is essential not to convert the numbers to decimals, to carry out operations decimally and only to change results back to sexagesimals . . . roundings in one system do not mean the same in the other and accurate parameters given in sexagesimals may be altered by the transition through decimal computations.
>
> Otto Neugebauer, *A History of Mathematical Astronomy* (1975:1113)

Thus Neugebauer's unfathomable rejection of circular motion and further insistence on sexagesimal mathematics for Babylonian astronomy. But his earlier closing remark concerning mathematical tools was not only incorrect, it was also decidedly unwise. In fact, it is the height of irony that around the time HAMA was published (1975), Robert R. Newton, using a 10-decimal place pocket calculator put an end to the entirely unmerited reign of the *Almagest* by showing that the latter's supposedly accurate daily velocities for the planets (each given by Ptolemy to the *sixth* sexagesimal place) were *all erroneous*.[45] And therefore, they no longer "saved the phenomena," as claimed, while Otto Neugebauer, having spent his illustrious career heaping praise on Claudius Ptolemy and the *Almagest*, was obliged to admit in a feeble footnote in HAMA that he too had never checked the accuracy of the daily velocities in the chronically flawed *Almagest*.[46] More to the immediate point, it also turns out that the velocities so used were in all likelihood appropriated from limited, *earlier* sets of completely misunderstood Babylonian period relations.

It could be said that awareness of these errors only came to light with the publication of Robert R. Newton's *The Crime of Claudius Ptolemy* (1977)[45] and his *The Origins of Ptolemy's Astronomical Parameters* (1982),[47] but this is not entirely so. Both Al-Bitruji (ca. 1200)[48] and Copernicus (ca.1500),[49] using for the most part the same period relations as Ptolemy were also unable to obtain compatible results, although the former, with a liberal sprinkling of the word "about," at least forewarns the reader of the uncertainties involved.

The main difficulty, it would seem, is not so much *geocentricity*, but restrictions imposed by having to incorporate *uniform circular motion* in association with basically *unworkable* auxiliary devices. Plus one further restriction - *also never met with Ptolemy's values* - which was that the sums of the supposed velocities in "anomaly" and "longitude" in each and every case were required to *equal* the velocity of the Sun.

One can certainly work though all the period relations employed by Ptolemy, Al-Bitruji and Copernicus to check the disparities discussed by Robert R. Newton, although it is nevertheless an irritating task. At present it is sufficient to note that as recently as the second half of the Twentieth Century - Manitius[50] (1963 reprint), Pedersen (1974)[51] and even Neugebauer (1975)[52] were unaware of the multiple errors in the daily velocities in Ptolemy's *Almagest*.

Perhaps Neugebauer was encouraged and emboldened by lack of criticism for earlier omissions in ACT (1955), not to mention its unreadable and disjointed presentation, but for whatever the reasons, even passing knowledge of the Babylonian procedures in the astronomical cuneiform texts of the Seleucid Era - the primary source for both ACT(1955) and HAMA (1975) - indicates that the Babylonian methodology routinely and indisputably employed uniform circular motion for the *initial* derivation of the *mean* synodic arcs for Sun, Moon and the planets. But unlike the Ptolemaic planetary model which utilized uniform circular motion and auxiliary devices to account for observed variations, the Babylonian used *mean circular motion* for the *mean synodic arcs* as the basis for schemes to account for variations *on either side* of established mean values. Circular motion is certainly applied in Babylonian astronomy but as *an intermediate step* towards the determination of apsidal velocities (predominantly System A) with more detailed schemes to determine the *varying velocities* and the associated *varying times* in System B. All of which was achieved before the time of Hipparchus (*ca*.150 BCE) and well before Claudius Ptolemy's much-praised *Almagest*. Furthermore, the same fractional parts of misunderstood Babylonian period relations were, it would seem, not only applied by the latter, but also used in the alternate *geocentric* planetary model by Al-Bitruji (ca.1200) and ultimately in the *heliocentric* model proposed by Nicholas Copernicus (*ca*.1500).

VIII. Babylonian astronomy during the 20th Century

Then how did the current mindset which holds that the Babylonians themselves possessed no planetary model or any fictive understanding become established? There are numerous issues in ACT for which corrections are long overdue with unknown parameters and operations in the Babylonian procedure texts in need of explanation. Some deficiencies are elementary enough, but they should surely have been fully addressed by now, specially for Jupiter. This is clear from the distribution of the planetary texts in ACT cataloged by Neugebauer in his Introduction to the Planetary Theory in General: [53]

> Our material contains ephemerides for all five planets. The distribution is very uneven: 41 texts and fragments for Jupiter, but only 12 for Saturn, 11 for Mercury, 9 for Venus, and 8 for Mars. In quoting these texts, we adopt the the following notation:
>
> Nos. 300 ff. Mercury
> Nos. 400 ff. Venus
> Nos. 500 ff. Mars
> Nos. 600 ff. Jupiter
> Nos. 700 ff. Saturn
> Nos. 800 ff. Procedure texts
>
> The "procedure texts" also deal with all five planets but here, too, Jupiter is better represented than all the other planets together.
> The foundation of our understanding of the planetary texts was laid by Kugler[1] and Pannekeok.[2] Later contributions were due to Schnabel,[3] and van der Waerden.[4] Our material seems to be large enough to justify the statement that the planetary theory was less developed than the theory of the moon. We find, *e.g*., no direct consideration of the latitude. The complications of the planetary motion on one hand, the lack of of calendaric importance on the other might be the reasons for this fact.
> It must not be forgotten, however, that our knowledge of Babylonian planetary theory rests on a much smaller basis than the lunar theory. Not only is the total number of texts for all five planets less than half of the lunar texts, but also the state of preservation of the procedure texts is much worse in the case of the planets than for the moon.
>
> Otto Neugebauer, *Astronomical Cuneiform Texts* (1975:279)

The inclusion of "procedure" texts is an initial surprise, as is the larger luni-solar component with its potential for providing insights concerning the Babylonian approach to Earth/Sun dualities. This is the essential matter that lies at the heart and soul of all Solar System planetary theories.

So far Neugebauer's introduction to the Babylonian material is encouraging as well as intriguing, and the same can be said concerning the cautions included in his explanation of "The Leading Ideas of the Planetary Theory." [54]

> The discussion of the Babylonian planetary theory which follows is not intended to give a historically or astronomic -ally complete description of this field. Its only purpose is to prepare the reader for the study of the texts themselves, that is, to enable him to compute the ephemerides with the methods used used by the ancients and to grasp the general trend in the procedure texts. For all details, however, he must consult the texts themselves plus the notes and commentaries added to the translations and the relevant remarks in the Introductions concerning the theory of the single planets. The reader must remember what I have said in the preface: this is only an *edition* of the texts, not a systematic study of the contents.
>
> Otto Neugebauer, Astronomical *Cuneiform Texts* (1955:279)

But if ACT (1955) was indeed "only an edition of the texts, not a systematic study of the contents," then without further analysis, how did the notion that this material was insufficient to support a fictive planetary model arise? A partial answer would seem to be that p*rior analytics* with initial emphases on available tools - units of time and measure, astronomical reference frames, mathematical methodology and results as recorded in the astronomical cuneiform texts - were not carried out adequately. Yet the latter seem to be relatively limited sets of data, *i.e.,* the cumulative progress of the planets in terms of varying synodic arcs with accompanying synodic times, in short, to[55]

> find the longitudes and dates for the consecutive "*characteristic phenomena*" of the planetary appearances, such as helical risings and settings, oppositions, stations, etc. Only if these phenomena are known, can the location of the planet be determined for intermediate moments by means of complicated interpolations. But the overwhelming majority of the "ephemerides" and the procedure texts are devoted to the problems of determining the sequence of the characteristic phenomena, whereas the problem of "daily motion" is of a secondary character.
>
> Otto Neugebauer, *Astronomical Cuneiform Texts* (1955:279)

The "secondary" nature of daily motion is not a problem if it is given due consideration. It does, moreover, supply a reminder that *the diurnal axial rotation of Earth* necessarily plays a fundamental role in the various "risings," "settings," "disappearances/last visibilities in the West" and "appearances/first visibilities in the East" of the synodic phenomena. Not alone, however, but with the daily *rotation* of Earth about its axis while simultaneously completing its annual orbital *revolution* about a common center, and in the same way the longitudinal progress of each planet. But it is here that Neugebauer's description of Babylonian synodic phenomena takes a decidedly strange turn. First of all, although he is willing to concede that the Babylonian approach "is obviously the 'natural ' one," he nevertheless follows this reasonable statement with an historical aside that negates it while elevating later Greek achievements. Which in the case of Ptolemy and the vaunted *Almagest* turns out to be quite unmerited. Thus in full Neugebauer asserts that: [55]

> The Babylonian approach is obviously the 'natural' one.' What one realizes first about the planets is their appearances and disappearances in the nightly sky, their stations and retrogradations. To predict these phenomena seems to be a real problem and it was solved by our texts by means of very ingenious arithmetical devices. But it marks an enormous step forward to ignore the 'natural' problems altogether and to ask an apparently much more complex question: how to describe the planetary motion as a whole. It is this shift of emphasis which led Apollonius, Hipparchus, and Ptolemy to their enormous successes.
>
> Otto Neugebauer, *Astronomical Cuneiform Texts* (1955:280)

Unfortunately, most new readers would accept this assessment without question, coming as is does from noted Historian of Science Neugebauer who is also the foremost authority on Babylonian astronomy. Even so, from what he has said earlier about the limited and specialized nature of the Babylonian approach it would be premature to embark on such comparisons. But more to the point, despite Neugebauer's proclamations and denigrations, the instructions to his readers to consult the contents of the Babylonian texts provides the means to test whether the Babylonian methodology does (or does not) describe "planetary motion as a whole." And also - a necessary aside- why would anyone conclude that it did not, with the form of their approach of secondary importance, even if it is almost self-evidently *heliocentric* ?

The main problem, it seems, is that Neugebauer not only treated the Babylonian materials in the wrong order, he also failed to provide the necessary groundwork and the requisite tools for the job. Furthermore, commencing with the relatively complex motions of the moon followed by a convoluted treatment of the motions of the two Inferior planets Venus and Mercury was premature in the absence of a coherent treatment of lunar and planetary motion. Which is not to make light of such requirements, even though matters are helped to a considerable extent by the application of essentially the same synodic formulas in both lunar and planetary contexts. Which is hardly surprising in view of the name allotted to the commonly observed period for the moon, *i.e*., the "synodic" month, simplistically, from one full moon to the next. But what is not simple, however, is that although there is only one moon orbiting Earth, its motions occur at the same time that the latter - while concurrently rotating daily about its axis - is also orbiting the Sun and completing one full revolution of 360° in one year. And once again, there is more than one kind of month and more than one kind of year, not to mention additional inter-related cycles with periodic variations caused by the elliptical natures of the various orbits, inclinations and precession, etc. Plus the fact that the observing platform (Earth) is subject to further variations caused by the considerable tilt of its own rotational axis.

Lastly, it should be noted that so far the present inquiry has been limited to the mean values alone. Omitted so far, the Babylonian treatment of varying velocities encountered in both luni-solar and planetary contexts is far more complex. Again this is *best understood in heliocentric terms*, current negative views on the matter notwithstanding. So, although not wishing to "concatenate without abruption," (as Dr. Johnson was wont put it) it is necessary to now include the Babylonian varying velocities in their various forms plus matters of related interest. But how could this be realized in the face of Neugebauer's bald statement that his analysis of Babylonian astronomy: [37]

> *will reveal to us the working of a theoretical astronomy* which operates *without any model of a spherical universe, without circular motions and all the other concepts which seemed "a priori " necessary for the investigation of celestial phenomena.*

The answer is simple enough. Easily, based on applied Babylonian synodic data and details in the "procedure" texts for Jupiter and Saturn, with both also employing the afore-denied "circular motions," and orbital progress observed, recorded and applied in Babylonian Systems A and B, where further surprises await.

IX. Babylonian Systems A and B synodic arcs and times; elliptical orbits

So far the discussion of the parameters of Systems A and B has been largely concentrated on the apsidal synodic arcs of System A, the mean synodic arcs of System B, an intermediate mean arc from System A' and the unattested theoretical mean values which resulted in Fibonacci ratios of 8/5, 5/3, 8/3, 144/55, 55/144 and 144/89. Yet there are unfortunately difficulties with System B as Aaboe's 1964 analysis of BM 37089 demonstrated which can be traced to a lack of awareness of the underlying formula for the determination of the extremal arcs of System B. But this itself is not the only issue since a secondary part of the problem also involves the role played by the mean synodic month (MSM = 29;31,50,8,20^{d}) in Babylonian planetary theory. It is here, however, that the choice of the slightly too high (but convenient) sidereal year MYR of 12;22,8 mean synodic months (or SYR = 365;15,38,17,44,26,40^{d}) becomes clear, especially when used with the 360° Zodiac's 12 x 30° zodiacal months and "tithis," *i.e.,* thirtieths of the mean synodic month. In other words, rather than determine the time required for Jupiter to move the mean and varying synodic arcs using the standard year in days and mean synodic months (SYR and MYR above) parts of the work were done using thirtieths, which is a major simplification for a base-60 numerical system. And with pre-selected parameters, also finite, which is where the rounding of the mean synodic arc for Jupiter of (u) = 33;8,45° (from 33;8,44,48,29,..°) unfortunately also complicates matters, at least for those unfamiliar with the methodology.

The reason for this situation lies in the fact that while Babylonian "procedure" text ACT No. 812 provides a method for the mean synodic time for Jupiter in the badly damaged first section and an alternate approach is given in the second, the two methods yield different answers. Or more precisely, results given to *three* and *five* sexagesimal places respectively - "13;30,27,46" in the first section[56] and "13;30,27,46,16,40" deducible from the conclusion of the second. Now *had this value* been given in full in the latter it is safe to say that Neugebauer (who was left "completely in the dark" by the three-place version in the first section) would almost certainly have realised that both parameters were the mean synodic period of *Jupiter expressed in mean synodic months* with three-place 13;30,27,46 MSM the value applied in the first section.[57]

An unfortunate consequence of the damaged state of Section1 of ACT 812 is that already condensed details are further obscured, but the mention of an alternate approach in Section 2 suggests that the first method was merely the application of the Babylonian integer period relation for Jupiter, essentially, where MYR = 12;22,8 MSM:

36 revolutions (Z) and 391 synodic periods or arcs (II) in 427 (N) years (4)

with the time for the mean synodic cycle simply 427/391•MYR = 13;30,27,*45,52*, . . MSM necessarily rounded up at the fourth sexagesimal place to 13;30,27,46 MSM (398.89078921. . . days). Plus, by the same methodology and the integer long period relation for Saturn:

9 revolutions (Z) and 256 synodic periods or arcs (II) in 265 (N) years (5)

the mean synodic time for Saturn is: 265/256•MSM = 12;48,13,26,15 MSM (378.10183198 days) with both methods producing identical answers because no rounding is required for the mean synodic arc of Saturn (u =12;39,22,30^{o}). From this viewpoint the duality is best demonstrated by the Saturn integer period relation, but there is also a more critical matter involving the first section of ACT 812 to be addressed, which is the determination of the extremal or apsidal synodic arcs from the integer period relations. Which, despite the damaged state of Section 1, is readily_ apparent from what remains. Neugebauer's transcription and commentary of No. 812 Section 1 plus one addition to line 6 is given below. The *15 pa* in line 5 is the location of the mean value (u) at15 Sagittarius in the Zodiac; *15 zib* and *15 absin* line 6 are the locations of the maximum (M) and minimum (m) arcs/times at 15 Pisces and 15 Virgo.[58]

No. 812

BM 34221 + BM 34299 + BM 35119 + BM 35206 + BM 35445 +BM 45702 (= Sp.327 + Sp.410 + Sp.II,664 + Sp.II,763 + Sp.II,1034 + SH 81-7-6,107)

Contents: Procedure text for Jupiter and Saturn
Arrangement: O/R
Provenance: Babylon [Sp. and SH]

Section 1

Obv. I, beginning destroyed

Transcription

1[. . . .] . . *lu*-DU-*ma*
2[. . . . .] 13,30,27,46 a-rá 6,31 DU-*ma*
3[. . . . .] 13,30,27,46 *qablū*(murub$_4$-u)
4[. . . . .] *tašakkan*(gar-*an*) 1,48 a-rá 6,31 DU-*ma*
5[. . . . . *ēpuš*(dù -]*uš*) 15 pa 45,14 u$_4$-meš
6[15 zib 50,7,1]5 15 absin 40,20,45 u$_4$-meš.

Commentary

It is certain that this section concerns System B of the theory of Jupiter. This is shown by the occurrence of the following constants (*cf.* Introduction p. 311):

Lines 2 and 4	6,31 = *II*	(= 391 mean synodic arcs/periods, added)
line 4	1;48 = d	*of* ΔB or ΔT
line 5	45;14^{r} = u	of ΔT
{line 6	50; 7,15^{r} = M	of ΔT (added)}
line 6	40;20,45^{r} = m	of ΔT

The value 50;7,15^{r} = M of ΔT is restored from the parallel passage of No. 813 Rev. I,5. The combination of these two sections shows that it was assumed that m corresponds to ♍ 15, M to ♓ 15 and consequently u to ♐15 and ♊15. In System A the apsidal line goes through ♍ 12;30 and ♓12;30.

Completely in the dark, however, remains the number . . .]13,30,27,46 in lines 2 and 3. In line 3 it seems to be called "mean value". The same number occurs in the parallel passage of No. 813 Section 12 but once broken into [13,30] at the end of of a line and 27,46 at the beginning of the next line. This may be either a real separation or simply split writing; no other passage shows a separation between 13,30 and 27,46. In No. 813 the number [13.30] 27,46 seems to be added to 45,14 which is the mean value of ΔT."

In line 4 the difference d = 1;48 is multiplied by the number period II = 6,31. The result (11,43;48) is not preserved but there is hardly space left for more than the number which would represent the total variation 2 (M - m)Z of ΔT. In line 2 the "mean value" (?) . .] 13,30,27,46 is multiplied by *II*, again for unknown reasons."

Otto Neugebauer, *Astronomical Cuneiform Texts,* ACT (1955:392-393).

Perhaps the most striking part of Neugebauer's analyses of ACT 812 Sections 1 and 2 is a puzzling inability to come to terms with what must surely be the elementary determination of the mean synodic period for Jupiter ***in mean synodic months*** followed by the same for the varying periods of System B. Plus, even though Section 1 is damaged, there is still sufficient information to understand precisely what the remaining parameters mean with confirmation found in the "alternate method" described in Section 2, which according to Neugebauer's translation teaches: [59]

> . . . the computation of the difference between synodic time and synodic arc (disregarding the trivial 12 months) following exactly formula (14), Introduction p. 286, for $i = 1$.
>
> > [Alternate (method): 33;8,]45, the mean(-value) of the longitudes, multiply by 0;1,50,40, and (you obtain)[1;8,8,]20. Add to it 11;4, and (you obtain) 12;5, 8, 8,20. Put it down for the gabari's of (one) year.
>
> The first number, 33;8,45^r, is the mean synodic arc $\Delta\lambda = u$ in Systems B and B'. The epact is $\epsilon = 11;4$ and consequently
> $$\frac{\epsilon}{6,0} = 0;1,50,40^r.$$
> Thus our text follows the formula $$u\frac{\epsilon}{6,0} + \epsilon = 12;5,8,820^r.$$
> The abbreviated value 12;5,8^r is known from System B' (cf. Introduction p. 311). A still more rounded-off value 12;5,10^r, is applied in the majority of the ephemerides. For the term gaba-ri *cf*. below, p. 413.
> What follows must mean that the value 12;5,8,8,20^r should be added to the synodic arc. Unfortunately, some of the corresponding signs are damaged, but one may perhaps read
>
> > [From (one) appear]ance to (the next) appearance, (the arc) between them you put down. [12;5,8,8,20 you add to it and predict the [dates]s.

X. Jupiter

Now had Neugebauer *carried out* the instructions he would have obtained 12;5,8,8,20 + 33;8,45 = 45;13,53,8,20 as opposed to the *mean* value "45;14^r = u of ΔT" with constants for the *maximum and minimum times* also included between lines 5 and 6 of Section 1. So just where was Neugebauer's problem? Apart from temporal racism, in the present case it can be traced to one line in the latter's translation and commentary for Section 2 of ACT 812 where he disregards "the trivial 12 months," except that elsewhere he stipulates that the 12 months should be added to to obtain the synodic times, which is indeed the last requirement to actually predict the dates. But one other earlier step, which is to convert back to days after carrying out key parts of the computation in "tithis" was not apparently carried out. As a result, his final representations of the synodic times for Jupiter using the above values remained in in mixed format. Thus 12 mean synodic months of 29;31,50,8,20 days to be added in the final step to values were instead combined by Neugebauer with results which were still in tithis yet still needed to be divided by 30 before the last addition of the 12 months. Therefore division of the end product in the text by 30, *i.e.*, 45;13,53,8,20/30 = 1;30,27,46,16,40 plus the final addition of *12 mean synodic months* results in a mean synodic period of Jupiter of 13;30,27,46,16,40 months (MSM). Which can then be compared to the three-place number 13;30,27,46 (MSM) in Section 1 - the unknown parameter which left Neugebauer "completely in the dark."

Once aware of the latter result, however, the application of the concept to the contents of lines 5 and 6 of Section 1 naturally follow, *i.e.*, subtraction of the mean synodic arc 33;8,45° (located at Sagittarius 15) from 45;14, the like subtraction of the maximum synodic arc M = 38;2° from 50;7,15 (located at Pisces 15) and a final subtraction of the minimum arc m = 28;15,30 from 40;20,45 located at Virgo 15 all result in the more convenient constant 12;5,15.

It turns out that there are four constants: the most accurate being that derived in Section 2 of ACT 812, the next 12;5,8, followed by 12;5,10 and lastly, the above value 12;5,15 as the least accurate but the most convenient to use. In actual fact, the differences for the final periods in days for all four constants are quite small, *e.g.*, using the mean synodic arc (u) for Jupiter of 33;8,45° and MSM (29;31,50,8,20 days), the periods in MSM and days are:

Jupiter k	k + 33;8,45	Divided by 30	+ 12 MSM	Synodic S (days)
12;5,8	45;13,53	1;30,27,46	13;30,27,46	398.890789
12;5,8,8,20	45;13,53,8,20	1;30,27,46.16.40	13;30,27,46,16,40	398.890827
12;5,10	45;13,55	1;30,27,50	13;30,27,50	398.891336
12;5,15	45;14	1;30,28	13;30,28	398.892703

Table 6. Babylonian Jupiter time constants and resulting periods (S) in days.

with the slightly low constant 12;5,8 now explained by its use to generate the "mystery" period 13;30,27,46 MSM.

More critically, however, Neugebauer does not explain that the multiplication factors in Section 2 originate from the relation which incorporates "tithis" (thirtieths of the mean synodic month, ^{r}), *i.e.*, 30•12;22,8 MSM = 371;4, with division by 360° providing a time per degree for the motion of either Earth (or Sun) of *1;1,50,40*. Which is where the simple addition of the mean synodic arc (u) 33;8,45° then (u)+N•d can be seen as a separate "multiplication" factor of 1x from 1;1,50,40 with the results recorded in a separate column as the first half of the task with the second the corresponding times from the standard constant(s) based on 0;1,50,40 or 0;1,50, etc.

XI. Determination of the Babylonian extremal synodic arcs/times and the varying values of System B

Up to a point this problem can be said to have already been dealt with by Neugebauer's detailed descriptions in *THE PLANETARY THEORY IN GENERAL* in *Astronomical Cuneiform Texts* (1955:284-287). Except for two fundamental issues, which is that no instructions were given by Neugebauer to generate a System B from scratch, despite his familiarity with the components necessary for this purpose. This is also why Aaboe (1964:32) was unable to develop a System B from the BM 37089 data for the Earth/Sun based on the difference d = 0; 0,1,32,42,13,20° per day, and although incomplete, the equally extensive daily velocities in the text. The second deep-seated problem is how any progress could be expected in the absence of a fictive approach to planetary motion and/or some form of model.

Returning to this central issue in the context of ACT 812 Section1 the method for establishing the apsidal synodic arcs becomes apparent from the residual parts of this text, specifically lines 2 and 4 with Neugebauer's comments in part also emphasizing the likely generation of the mean synodic time: [39]

> . . . Completely in the dark, however, remains the number . . .]13,30,27,46 in lines 2 and 3. In line 3 it seems to be called "mean value". The same number occurs in the parallel passage of No. 813 Section 12 . . . (where) the number [13.30] 27,46 seems to be added to 45,14 which is the mean value of ΔT."
>
> In line 4 the difference d = 1;48 is multiplied by the number period II = 6,31. The result (11,43;48) is not preserved but there is hardly space left for more than the number which would represent the total variation 2(M - m)Z of ΔT. In line 2 the "mean value" (?) . .] 13,30,27,46 is multiplied by II, again for unknown reasons."

Recalling again that the Babylonian long integer period relation for Jupiter is:

36 revolutions (Z) and 391 synodic periods or arcs (II) in 427 (N) years (4)

and now aware that the mystery number "13,30,27,46" is unequivocally the mean synodic period for Jupiter in mean synodic months it is clear that the multiplication of this constant by 391(the number of synodic periods for Jupiter) yields the total synodic motion over 427 years and 36 revolutions. Whereas the further multiplication of 391 by the difference d = 1.8 (391•1;48) yields a total of 11,43;48 (703.8). The division of the latter by 36 therefore provides the number of increments per synodic arc for one revolution = 19;33 (19.55) with the further division by two giving the amplitude = 9;46,30 (9.775). As in fact included by Neugebauer in his analysis of Babylonian System B for Jupiter where the subtraction of the minimum synodic arc (m) of 28;15,30° from the maximum (M) of 38;2° results in 9;46,30.

But this range - *aphelion to perihelion* - can be *halved* again to take advantage of the relation for the mean synodic arc (u) = ½ (m + M) whereas the parameter P is (391/36) and the full amplitude also Pd = 19.55. Therefore, with the inclusion of the mean synodic arc (u) the maximum (M) and minimum (m) synodic arcs for System B can be derived from the general relation:

Babylonian System B extremal synodic arcs (M, m) = u ± ¼Pd (6)

From this viewpoint the period T is dependent on the mean synodic arc (u) with the only variable available d, the amount which increases from aphelion to perihelion and *vice versa* from the latter back to aphelion, etc.

XII. System B locations and times for the extremal/apsidal synodic arcs of Jupiter

Based on the integer period relation (4) above, the parameters of System B for Jupiter reduced to their simplest form are:

11;51,40 (11.86111*) years for the period of revolution T (427/36)
10;51,40 (10.86111*) years for the parameter P (391/36, 360/u) and/or T - 1)
1;48° for the difference per synodic arc d (*variable*(?), according to choice)

while relation (6) provides the extremal velocities (m) and (M) which, along with the mean synodic arc (u) below completes the set:

28;15,30° for the minimum synodic arc (m)
33; 8.45° for the mean synodic arc (u)
38;2° for the maximum synodic arc (M)

Corresponding times - as shown in Table 14 - are dependent on the choice of constant (*i.e.*,12; 5,8 through 12;5,15).

However, System B' used in ephemeris ACT No. 640 and procedure text ACT No. 813 Sections 21 and 22 [60] utilize a value for d of 1;46,40 instead of the standard value 1;48 which, with u and P remaining constant, results in new values for M and m based on ¼Pd = 4;49,37,46,40 with extremal velocities of 28;19,7,13,20° and 37;58,22,46,40°. All three may be compared to the equivalents given by Neugebauer of 9;39,10° (the half = 4;49,35 is not included by the latter) followed by 28;19,10° and 37;58,20° deduced from the above texts. Neugebauer's treatment of this variant includes both a change in the underlying period while still maintaining the standard mean synodic arc of 33;8,45°. It also contains "certain discrepancies . . .which are not exclusively due to rounding-off for the sake of easier computation." These include the observation that replacement of the constant 12;5,8,8,20 for the mean synodic time with the lesser value 12;5,8 is "a permissible approximation." And certainly, rounding applied to the four-place data to reduce it to two-places is indeed reasonable. Yet there remains a problem since Neugebauer's constants for the positions and times also vary. This is no small matter, although it does serve to emphasize that the methodology, - despite numerous variants and details - was not fully understood, especially when *tithis and the mean synodic month* are involved, as in the present case.

More to the point, Neugebauer also provides the following commentary concerning a "third method" supplied in Sections 14,15 and 16 of ACT 813 for System A' oddly exemplified by the use of much rounded parameters for the velocity of the "Sun." Thus he writes:

> In spite of many difficulties in the reading of these sections, which are headed ' "third (method)", the main contents can be established. We are dealing with the slow, medium, and fast arcs of System A'. The synodic arcs 30 and and 33;45 are given, and the corresponding synodic times are to be found. In order to travel one degree the mean sun requires $\frac{1\,;\,0{,}57{,}13}{0\,;\,59{,}8}$ = 1,50,49, . . ʳ. If we abbreviate this value we obtain for a synodic arc of *w*° a time of *w*+ 0;1,50*w*.
> We know furthermore that 12 lunar months are 11;3,20 ʳ shorter than one solar year (lines 10, 11, 15). Thus the time difference in question is found in Section 15 as follows: (added: zodiacal names included for the signs)
>
> > From 9 Cancer (*sic!* for Virgo) to 2 Capricorn (from) year to year 33;45. . . (from) year to year 33;4[5 . . . .] multiply by 0;1,50 and (the result) 1;1,52,30 add to 33;45. 34;46,52,[30] and 11;3,20, the days of the sun, add together and 45;50,[12.30 put down for the gaba-ri].
>
> In Section 16 the factor 0;1,50 appears in the form 0;2 - 0;0,10:
>
> > From 2 Capricorn to 17 Taurus (from) year to year 36 . . . . [. . .] . . . . [. . . . .] . . . . . 36 . . . . multiply by 0;2 and [(the result) 1;12 . . . . 36 multiply by 0;0,10 and] subtract 0;6 from 1;12 and what remains (namely) 1;6 [add to 36 and to 11;3,20. The result is 48;9,20
>
> In Section 14 one should expect for the slow arc a time difference of 30•1;1,50 = 30;55 whereas the text gives only 30;45:
>
> > From 9 Cancer to 9 Scorpio] 30 (degrees or 1) danna and (from) year to year 11;3,20 days subtracted(?) and (?) 30 (degrees or 1) dan[na . . . . . .] . . . of Jupiter (from) year to year . . . 30;45 days multiply(?) and 30;45 days [and 11;3,20] days add together and 41;28,20 days put down for the gaba-ri of (one) year. Neugebauer, *Astronomical Cuneiform Texts,* ACT (1955:392-412).

Before discussing the above and the accompanying implications, there is, however a more immediate issue which concerns Neugebauer's introduction above of the ratio "1;0,57,13 / 0;59,8" and an approximate result of "1,50,39…" This is a mystery, since in Section 13 just before (which is parallel to Section 2), his translation included 0;1,50,40 *en route* to the determination of the constants for 33;8,45 (Jupiter (*u*), *i.e.*, intermediate values to be added (1;1,8,8,20 and 11;4) to obtain the final value 12;5,8,8,20. This is followed (as in the parallel text in Section 2) with condensed instructions for its use, which are worth repeating:

> Alternate (method): 33;8,45, the mean(value) of the longitudes, multiply by 0;1,50,40, and (you obtain) 12;8,8,20]. Add to it 11;4, and (you obtain) 12;5, 8, 8,20. Put it down for the gaba-ri of (one) year. The arc [between them you put down. 12;5,8,8,20 add to it and predict the dates. 1,1,8,820 . .]

The reason behind labouring this point is that there seems to have been insufficient understanding of the central role played by the inter-related trio of luni-solar constants applied in Babylonian astronomy, namely, with assigned abbreviations introduced earlier, the following:

1. Mean Synodic Month (MSM) = 29;31,50,8,20 days (29.530594135802474)
2. Sidereal Year in MSM (MYR) = 12;22,8 Mean Synodic Months (12.36888*)
3. Sidereal Year in Days (SYR) = SYR = 365;15,38,17;44,26,40 days (365.260637688614546)

to which can be added the Zodiacal "Year" of 360° with its 12•30° months of the Zodiac and the associated relations:

MSM•MYR = SYR (MYR = 12;22,8 MSM)
30•MYR = 371;4 (6,11;4 = 371.10666* *tithi*)

Thus arriving at the multiplication factor for the conversion *to tithi per degree* of *1;1,50,40* as used in the texts, along with a neat and simple method for concurrently recording the varying synodic arcs and their corresponding times.

$$\frac{30\bullet MYR}{360} = \frac{MYR}{12} = \frac{6{,}11{;}4}{6{,}0} = \mathbf{1;1,50.40}\ (1.03074074074^*)$$

Which is to *bifurcate* the multiplication factor **1;1,50.40,** *i.e,* first multiply the synodic arcs by the integer 1, then set this aside (or record) the "result" as the arc under consideration for subsequent positional assignment(s) in degrees, remembering that the conversion to tithis will later require conversion to days. Meanwhile, with unity (1) removed the remainder of the multiplication factor (0;1,50,40) can be applied to the arc itself. Thus essentially an operation in five stages:

(1) Synodic arc•1;1,50,40 for the arc with or without the difference (s) *d.*
(2) Syndic arc•0;1,50,40 + k = 11;4 (or 11;3,20)
(3) Step (1) combined with Step 2.
(4) (Step (3) sum divided by 30 to convert to MSM
(5) 12 MSM added to Step 4 for the corresponding Time in MSM

with, for the second part, the simpler two-place constant 0;1,50 as used in ACT 813, Sections 14 to 16.

Except that instead of MYR = 12;22,8 MSM the length of the year based on the replacement of 11;4 by 11;3,20 is 12;22,6,40 (12.36851851851852 MSM), the total in tithis is 371;3,20 (371.0555*) and the multiplication factor now 1;1,50,33,20. Which further explains why the later was reduced to 01;1,50 while also retaining the 1X factor for the for the synodic arc.

XIII. Sections 14, 15 and 16 of ACT 813 revisited

The major point of interest is that the 30°, 33;75° and 36° synodic arcs of System A' examined earlier with respect to the *Fibonacci* ratios found in the variants of relations 7, 8 and 12 are here assigned specified sectors in the Zodiac:

30° Minimum arc (*m*) assignment: 9 Cancer ♋ to 9 Scorpio ♏.
33;45° intermediate arc (u_2) assignment: 9 Scorpio ♏ to 2 Capricorn ♑
36° Maximum arc (*M*) assignment: 2 Capricorn ♑ to 17 Taurus ♉

with a line of apsides from 12;30 Virgo ♍ to 12;30 Pisces ♓ according to Neugebauer and Act 814, Section 2.[61]

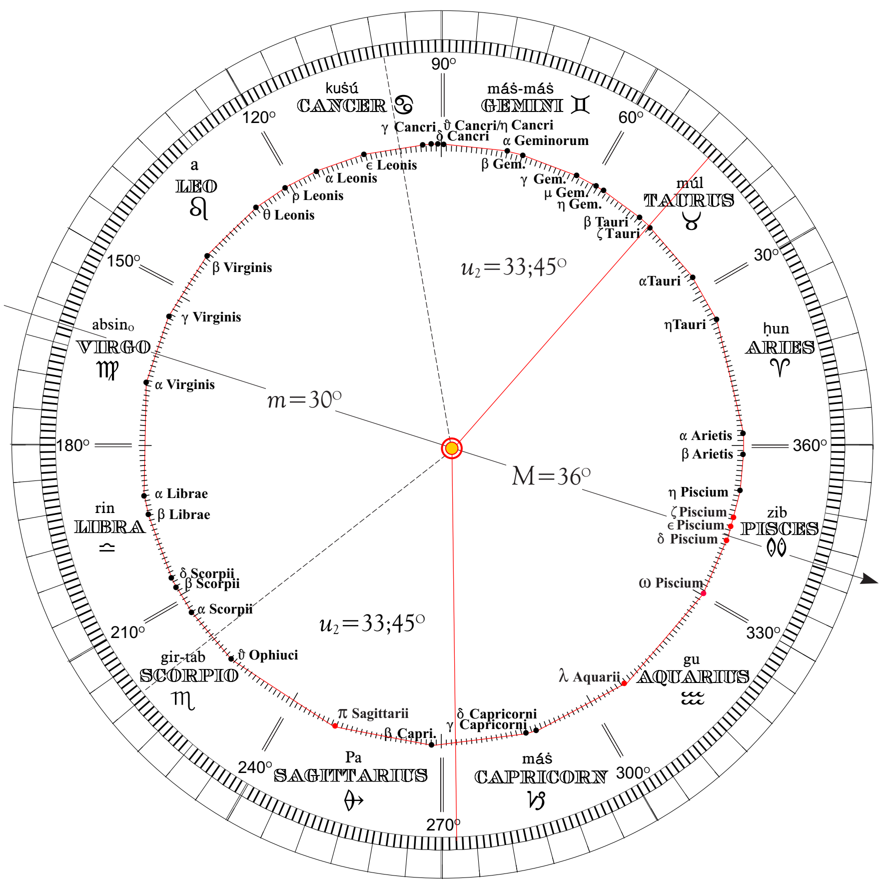


Fig 8. Jupiter System A' assigned sectors for the 30°, 33;45° and 36° synodic arcs

At which point it seems necessary to investigate this matter further in terms of elliptical orbits in general in light of the above assignments and previous discussions concerning the Babylonian synodic arcs.

Before returning to the ellipse in Babylonian contexts mentioned earlier, it may be helpful for the casual reader to revisit Kepler's three laws of planetary motion described and qualified by Cesare Emiliani (1995) as follows:[62]

> "FIRST LAW: ' The orbit of a planet is an ellipse with the Sun at one of the two foci.'
> This law establishes the shape of planetary orbits. They are not circles, but ellipses.
>
> SECOND LAW: ' A planet revolves around the Sun with the connecting lines sweeping equal areas in equal times.'
> This law establishes that the planets move faster when they are closer to the Sun and more slowly when they are farther away.
>
> THIRD LAW: ' The square of the sidereal period of a planet is proportional to the cube of the semimajor axis of it's orbit.'
> This law establishes that the sidereal period of the planets (the time it takes to go around the Sun once) increases as the distance from the Sun increases."
>
> Cesare Emiliani, *THE SCIENTIFIC COMPANION*, (1995:121)

For ellipses in the present context modern textbooks generally provide the mean distances *and* eccentricities in *a priori* formulas for the calculation of apsidal data, but what about the earlier Babylonian methodology ? As it so happens an alternative approach to the subject is provided in Robin S. Green's description of *"Kepler's equation for a bound orbit,"* published in *Spherical Astronomy* (1985), the relevant parts of which are as follows:[63]

". . . $F(r)$ is simply shorthand for the quadratic form given by

$$F(r) = r^2 - 2ar + h^2\, a/u. \qquad (6{:}19)$$

Equation (6:18) implies that $F(r)$ is ≤ 0. So we may conclude that the radius vector r is restricted to the range

$$r_1 \leq r \leq r_2. \qquad (6{:}20)$$

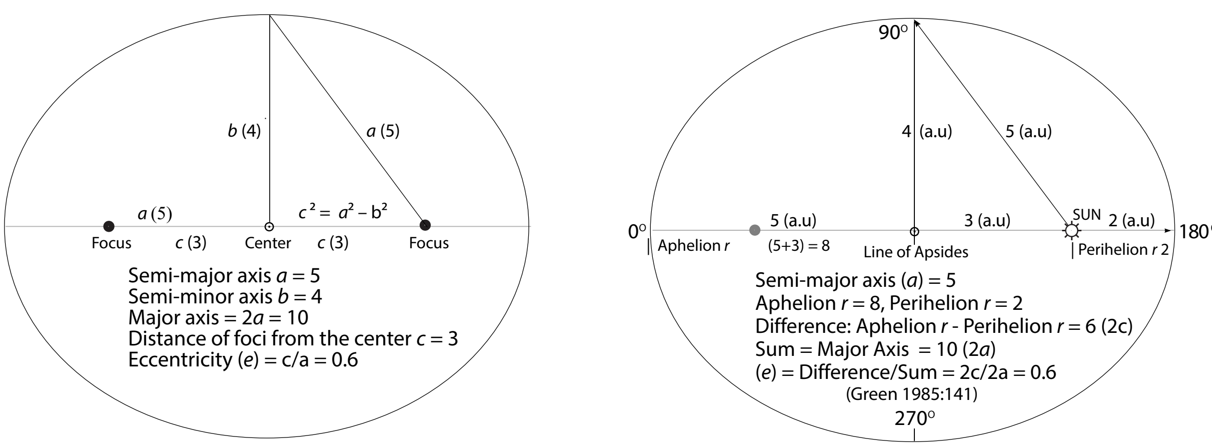
where r_1 and r_2 are the two roots of $F(r) = 0$. In fact, *they must correspond to the perihelion and aphelion distances.*
It should be noted, from equation (6:19), that $r_1 + r_2 = 2a$. Moreover, it is convenient to introduce the parameter *e - later to be identified with the orbital eccentricity* - as

$$e = \frac{r_2 - r_1}{r_2 + r_1} \qquad (6{:}21)$$

Then, working in terms of a and e, it is seen that $r_1 = a(1 - e)$, $r_2 = a(1 + e)$ (6:22)

Thus in the present context, all that will be required is the conversion of the Babylonian extremal/apsidal synodic arcs from degrees to time (T), then (via the harmonic law) to distance (R) to conform with relations (6:20) and (6:21) for the eccentricity and the remaining parameters.

Finally, in terms of background information it may prove beneficial to re-evaluate Green's relation (6:21) applied to the well-known ellipse based on the 5-4-3 Pythagorean triple in both standard and astronomical contexts:



I. Ellipse based on the 5-4-3 Pythagorean triangle

II. The 5-4-3 ellipse in astronomical context

Fig 9. I, II. The ellipse based on the 5-4-3 Pythagorean set in both general and astronomical contexts

Here the integral relationship between the semi-major axis 5 (a), the semi-minor axis 4 (b) and 3, the parameter (c) is routinely expressed as $a^2 = b^2 + c^2$ with the major axis $2a = 10$, the minor axis $2b = 8$, the distance between the two foci = $2c$ and eccentricity (e) from $c/a = 0.6$.

XIV. Apsidal distances for the elliptical orbit of Jupiter

System A maximum synodic arc (W) of 36° and minimum synodic arc (w) of 30° of Jupiter can be applied to the following variant of relation (13): $P = 360°/u = T - 1,\ T = P + 1$ (13)
i.e., from $T = (P + 1)$ and :- *Heliocentric distances* $R = [(360°/\text{Synodic Arcs } W, w) + 1]^{2/3}$ (13T)
to generate (via the harmonic law or variant) the following apsidal data for Jupiter with reference to unity ("a.u.").

Perihelion distance R from (360°/36°) +1, $T = 11$, $R = 11^{2/3} = 4.9460874$ (a.u.)
Aphelion distance R from (360°/30°) +1, $T = 13$, $R = 13^{2/3} = 5.2374311$ (a.u.)
(e) = (Aph R - Perh R)/(Aph R + Perh R) = 0.05562721 (modern: 0.04849485)
(determination of eccentricity e, R. M. Green, *Spherical Astronomy* 1985:141) [61]

The same process applied to System B extremal synodic arcs for both Jupiter and Saturn results in lower correlation for the eccentricities, whereas System A extremal synodic arcs for Saturn prove to be the most useful.

XV. Apsidal distances for the elliptical orbit of Saturn

Saturn's System A apsidal synodic arcs generate corresponding *heliocentric distances* from the maximum *(W)* synodic arc 14;3,45° and (*w*), the minimum synodic arc 11;43,7,30° as follows, with rounded distances included for a more immediate Babylonian application:

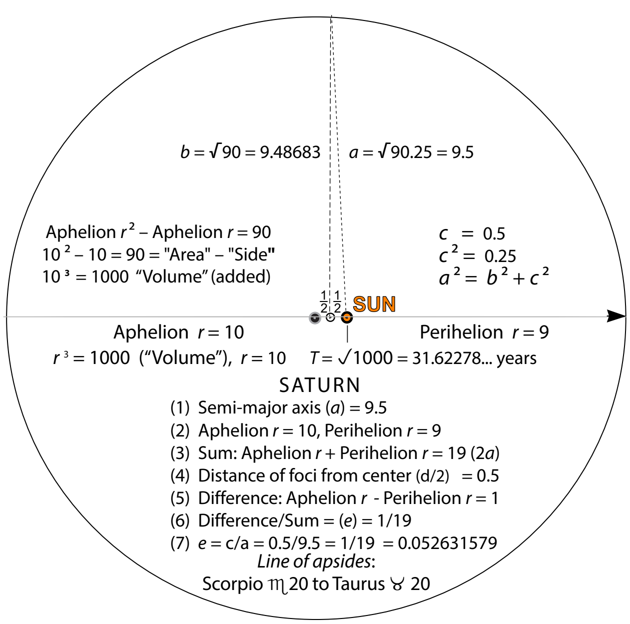
Perihelion R from (360/W)+1, $T = 26.60$, $R = T^{2/3} =$ 8.9108902 (a.u.), rounded: "9"
Aphelion R from (360/w)+1, $T = 31.72$, $R = T^{2/3} =$ 10.0204861 (a.u.), rounded: "10"
Mean Distance (265/9, Mean period, T, $R = T^{2/3} =$ 9.5353267 (a.u.) rounded: "9.5"
Distance between foci (d) = 1.109595829, c = 0.5547979145, rounded: "1" & "1/2"
(e = 0.0586115; rounded e = 1/19 = 0.0526316. modern e = 0.0555086)

All of which come into finer focus when applied to a Babylonian mathematical cuneiform text of hitherto unknown practical significance. As described by Marcus du Sautoy,[64] the text involves a procedure known as "completing the square," which - as du Sautoy notes - is equivalent to solving the quadratic equation $x^2 - x = 870$.

The latter's description of the text is given next in decimals for clarity with apsidal distances for Uranus similarly assigned to the right of the original text and the rounded data for Saturn applied next as the working example.

I have subtracted the side of my square from the area: 870.
You write down 1, the coefficient.
You must break off half of 1.
0.5 and 0.5 you multiply. You add 0.25 to 870. Result 870.25.
This is the square of side 29.5.
You add 0.5, which you multiplied, to 29.5.
Result 30, the side of the square. [ORIGINAL text]
[Quadratic equation : $x^2 - x = 870$]

I have subtracted the side of my square from the area: 380.
You write down 1, the coefficient.
You must break off half of 1.
0.5 and 0.5 you multiply. You add 0.25 to 380. Result 380.25.
This is the square of side 19.5.
You add 0.5, which you multiplied, to 19.5.
Result 20, the side of the square. [URANUS (added)]
[Quadratic equation : $x^2 - x = 380$]



Theoretical elliptical orbit for "Saturn"

The original Babylonian Mathematical Problem

I have subtracted the side of my square from the area: 870.
You write down 1, the coefficient.
You must break off half of 1.
0.5 and 0.5 you multiply.
You add 0.25 to 870. Result: 870.25
This is the square of side 29.5.
You add 0.5, which you multiplied, to 29.5
Result 30, the side of the square."

Method applied to the heliocentric elliptical orbit of **Saturn** based on rounded Babylonian System A data.

(Added : Quadratic equation : $x^2 - x = 90$; eccentricity e = 1/19)
I have subtracted the side of my square from the area: $90 = b^2$

You write down 1, the coefficient.	$d^* = 1$
You must break off half of 1.	$\frac{1}{2}d = c = 0.5$
0.5 and 0.5 you multiply.	$c^2 = 0.25$
You add 0.25 to 90. Result: 90.25	$c^2 + b^2 = 90.25$
This is the square of side 9.5.	From $a = \sqrt{(b^2 + c^2)}$
You add 0.5, which you multiplied, to 9.5	(c+a) =10, aphelion r
Result 10, the side of the square.	Saturn, *aphelion r*
also, (added) Subtract 0.5 from 9.5 = 9	Saturn, *perihelion r*

Saturn, mean distance : $\boldsymbol{R}$ = **9.5** (relative to unity)
*d = difference between *aphelion* and *perihelion* distances

Fig 10. The Babylonian mathematical problem and elliptical orbit of Saturn from rounded Babylonian System A data.

Similarly applied, with fixed "coefficients," $d = 1$, $c = 0.5$, the data for Uranus and Neptune are therefore.

Uranus: ($b^2 = 380$), add 0.5 to 19.5 ($c + a$) = 20, *aphelion r*
(Added: Quadratic equation : $x^2 - x = 380$; eccentricity $e = 1/39$)
(Added), subtract 0.5 from 19.5 ($a - c$) = 19, *perihelion r*
Uranus, mean distance : ***R*** = **19.5** (relative to unity)

Neptune: ($b^2 = 870$), add 0.5 to 29.5 ($c + a$) = 30, *aphelion r*
(Quadratic equation : $x^2 - x = 870$, the original problem ; $e = 1/59$)
(Added), subtract 0.5 from 29.5 ($a - c$) = 29, *Perihelion r*
Neptune, mean distance : ***R*** = **29.5** (relative to unity)

XVI. Uranus and Neptune

The two theoretical positions exterior to Saturn require that the "coefficient" *d* (1) and its half *c* remain constant, thus the eccentricities diminish with distance commencing with "sides of squares" of 10 (as used), 20 (unattested, but sequential), and 30, again sequential and the original target. In other words, the aphelion distances of 20 for Uranus ($e = 1/29$) and 30 a.u. for Neptune ($e = 1/59$). Lastly, on a more practical note, unaided detection of Uranus cannot be ruled out in any case; the latter, although faint is undoubtedly visible to the naked eye.[65-67] Alternatively, if the 2-(3)-5 Fibonacci series is theoretically assigned to the periods of Saturn, Jupiter-Saturn synodic *SD*1 and Jupiter, then the series will continue outward for two more planetary positions before ultimately ending at unity.

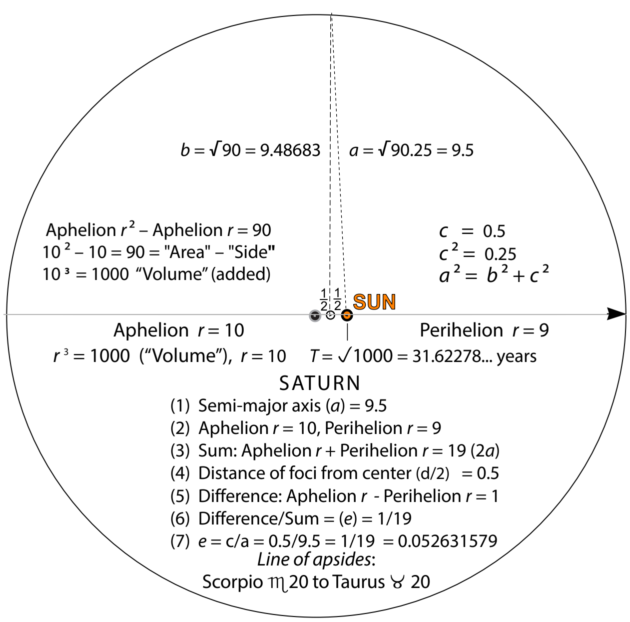
Thus commencing with the period of Neptune (1), Uranus-Neptune synodic also1 and the period of Uranus 2 the sequence is followed next by 1-(2)-3 to include the latter. Continuing inwards with the resonant Saturn-Jupiter triple [2-(3)-5] the sequence then proceeds in due order with 3-(5)-8, 5-(8)-13, 8-(13)-21, 13-(21)-34 and 21-(34)-55, etc. Thus essentially the inclusion of the intermediate synodic cycles to complete the Pierce-Agassiz model which finally became the extended *Phi*-series planetary framework.

Moreover, though wandering far afield, the above requirement that the "coefficient" *d* remains *unity* (1) suggests that certain "Pythagorean" matters involving the number 216 in Vitruvius' *Ten Books of Architecture* might have some bearing on the discussion. In particular, what may best be described as a "columnar" analogy, where the following three architectural "styles" are grouped together, as are the mean distances of ***9.5***, ***19.5*** and ***29.5*** for the Saturn, Uranus and Neptune ellipses assigned above, with the "coefficient" ($d = 1$) and half ($c = ½$) incorporated thus: [68]

> In the ***systyle,*** let the ***height*** be divided into ***nine and a half parts***, and ***one of these given to the thickness of the column..*** If the building is to be ***systyle*** and monotriglyphic, let the front of the temple if ***tetrastyle***, be divided into ***nineteen and a half parts;*** if ***hexastyle,*** into ***twenty-nine and a half parts***. (emphases supplied)
>
> *VITRUVIUS: The Ten Books on Architecture*, Bk III, Ch. III,10. (trans. M H. Morgan,1960:84)

XVII. Completing the Square

The application of rounded System A parameters for Saturn permits a comparable solution to the problem and also supplies a practical meaning which can be expanded to include *theoretical distances* for both Neptune *and* Uranus. This does not, of course, establish that either of the orbits of the outermost planets were known *per se,* but rather, that the suggested data can be interpreted as *theoretical* extensions beyond Saturn, the outermost planet known in Antiquity. As for "completing the square, this is readily demonstrated by dual figures for the Saturn data:



Theoretical elliptical orbit for "Saturn"

Fig. 11a. Saturn ellipse with rounded System A distances.

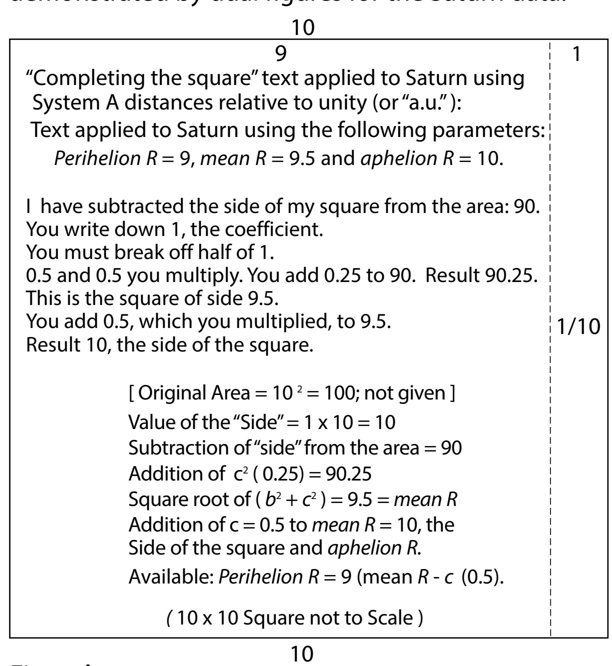


Fig. 11b. Completing the square procedure applied to Saturn.

Thus *aphelion distances* of 20 (a.u.) for Uranus and 30 (a.u.) for Neptune with the latter's mean distance of 29.5 a.u providing a B1 base period of 160.2260124 years and 164.3167673 years from the solution to the problem ("30"). All of which is sensibly heliocentric in both form and application with the mean heliocentric distances known and aphelion distances the target. Which also makes perfect sense, since the mean and aphelion distances are all that are required to supply the remaining parameters for practical ellipses with assigned lines of apsides. Therefore, if indeed astronomical in the above sense, then prior tasks from a modern viewpoint would entail the calculation of *cube roots* from the *squares of the periods of revolution* (*T*), thus "Kepler's" two-parameter law of planetary motion: $R = T^{2/3}$. As it turns out, the method used was identical in so much as the final product was still $R = T^{2/3}$ but includes the corresponding velocity with respect to unity (*V*r) and inverse (*V*i) exemplified by the Triple interval [$3^0, 3^1, 3^2, 3^3$] = [1, 3, 9, 27]. But before continuing in this vein the introduction of orbital velocities requires further expansion.

XVIII. Galileo, Plato, and orbital velocity in earlier times

Although including the time of Plato and latter's *Dialogues* some 400 years before the current era there is another matter which needs to be considered, which is orbital velocity in earlier times. Specifically, velocity expansions for the laws of planetary motion published in the *Journal of the Royal Astronomical Society of Canada* in a 1989 paper by the present scriber ("Projectiles, Parabolas and Velocity Expansions of the Laws of planetary motion") based on Galileo's treatment of orbital velocity in his *Dialogues Concerning Two New Sciences* (1638).[69] Predicated on Plato's *Timaeus* and *Epinomis* readers are invited to confirm Galileo's reasoning, when, after explaining his understandable hesitancy concerning open discussion of the subject, he states: [70]

> "but if anyone desires such information he can obtain it for himself from the theory set forth in the present treatment."

Which is how the present scriber - little more than a tertiary restorer - came to be involved in this matter. But either way the final products were assuredly an improvement over the two-parameter period (*T*) and distance (*R*) format currently in use. In other words, instead of the Harmonic or Kepler's Third law of planetary motion: $R^3 = T^2$, a series of additional formulas were now available incorporating *orbital velocity V*r (relative to unity) and *inverse Velocity* (*V*i). Thus the following velocity relations for the laws of planetary motion with related notes as published in 1989: [68]

$$"R = V_i^{\,2} \quad (1)$$

while the further relationships

$$V_r = R/T, \quad (2)$$

$$V_i = T/R, \quad (3)$$

$$T = V_i^{\,3}, \quad (4)$$

$$V_i^{\,6} = R^3 = T^2, \quad (5)$$

where T is the sidereal period in years, follow from Kepler's Third Law of planetary motion. Relation (5) may also be expressed in exponents (*i.e.*, [$V_i^0, V_i^1, V_i^2, V_i^3$]) and applied to the parabola as *the first three integer sets* which illustrate the Third Law.[7]

NOTES 5 and 7

5. Galileo discusses the sets [1,2,4,8] and [1,3,9,27] with respect to *squaring and cubing* in the *New Sciences,*(First Day [83]). The same sets are also mentioned by Plato in the *Timaeus* (35b and 43d), and the first set [1,2,4,8] is discussed again in *Epinomis* (991a-992a). Added, 2022: [1,1,1,1] (Earth), [1,2,4,8] (no body), [1,3,9,27] (Saturn, *perihelion V*i, *R, T*)

7. The ancient relationship between a point, a line, an area, and a volume. See Galileo's discussion of the latter pair and the "sesquialteral ratio" between them in the *Two New Sciences*, First Day, (134-135). "

The paper also included a comparison of the mean velocities of the planets utilizing additional ratios and also a constant velocity for Earth in kilometers per second to prove the value of the approach in practical terms.

MEAN PLANETARY DISTANCES, MEAN PERIODS AND MEAN VELOCITIES*

Planet	Distance (R)	Period (T)	$V_r\,(R/T)$	$V_a\,(kR/T)$	Modern
Mercury	0.387107	0.24085	1.60725	47.88	47.89
Venus	0.723350	0.61521	1.17577	35.03	35.03
Earth	1	1	1	29.79	29.79
Mars	1.523733	1.88089	0.81011	24.13	24.13
Jupiter	5.201287	11.86223	0.43847	13.06	13.06
Saturn	9.538188	29.45770	0.32379	9.65	9.64
Uranus	19.182303	84.01390	0.22832	6.80	6.81
Neptune	30.057937	164.79300	0.18240	5.43	5.43
Pluto	39.440188	247.69000	0.15923	4.74	4.74

* Table 1, "Velocity Expansions of the Laws of Planetary Motion," *JRASC*, Vol. 83, No. 3, p. 211, June 1989. Reproduced with permission of the Editor.

XV. The Double and Triple Intervals revisited;

The **Double** and **Triple** Intervals: [**1**, 2, **4**, **8**] and [**1**, 3, **9, 27**] are the first pair of integers which demonstrate the the *Harmonic Law* relative to unity: $R^3 = T^2$ (4 a.u.^{3} = 8 years^2 = 64, and 9 a.u.^{3} = 27 years^2 = 729). Plus, in the same sense, the "Single" interval [**1**, 1, **1**, **1**]] assigned to Earth (1 a.u.^{3} = 1 year^2 = 1) with the second parameter also providing the names for the intervals, but otherwise (in modern times) of unknown significance.

Surprisingly (and in whatever Babylonian context), the determination of distances relative to unity is nonetheless impressive in so much as it incorporates relative orbital velocity (*V*r) and inverse (*V*i) to streamline the process and simplify calculation. This much can be surmised from the Babylonian texts under discussion, augmented by the Pythagorean *Tetractys*, Plato's Double and Triple Interval plus Old Babylonian mathematical text VAT 8457 for the eventual calculation of cube roots from the periods of revolution (*T*).

Even so, there is one additional complication that arises from the simplicity of the final product which is that the corresponding heliocentric distance (*R*) is *not obtained from the cube root of the period squared ($R = \sqrt[3]{T^2}$) but the cube root of the period* (*T)* itself as the initial goal, *e.g.,* the *cube root* of 27 years is 3, which when squared yields the required distance (*R*) = 9. *What is the astronomical significance of the cube root of the period of revolution*?

Simply stated and denoted by ***Vi***, this value is *the inverse orbital velocity with respect to unity* corresponding to the *square root of the heliocentric distance* ***R*** and *the cube root of the period of revolution* ***T***. Which together, expressed in a format which extends beyond the two-parameter limitations of "Kepler's" Third Law ($R^3 = T^2$) is:

$$\text{Velocity } \boldsymbol{V}\text{i}^6 = \text{Distance } \boldsymbol{R}^3 = \text{Revolution } \boldsymbol{T}^2 \qquad \text{(relation 5, 1989)}$$

$$\text{Distance } \boldsymbol{R} = \text{Velocity } \boldsymbol{V}\text{i}^2 \qquad \text{(relation 1, 1989)}$$

with possible origins, it is suggested, demonstrated in Old Babylonian mathematical text VAT 8547.

XVI. VAT 8547 and the extraction of cube roots from periods of revolution

The present approach to the extraction of cube roots in VAT 8547 and similar texts differs from previous treatments which were handicapped by lack of context, whereas it is suggested here that they pertain to astronomy in general and *the cube roots of the periods* in particular. Which, as it so happens, *are also the square roots of the distances*, and therefore together they provide the basis (along with unity) for the "interval" concept, tetradic exponents, "Point Line-Square-Cube analogy" and "completing the cube" methodology. Even so, this does not explain the oddities in VAT 8547 discussed by Neugebauer and Sachs (1945),[71] Sachs (1952)[72] and Muroi (1989).[73] Nonetheless, what follows next is gained from these various commentaries, all linguistic complexities notwithstanding. Matters are, however simplified in so much as the same procedure is applied to all four numbers (or years), *i.e.,* 27, 64,125 and 216 with the procedure for the cube root of the first number (27) translated by Kazuo Muroi as follows:[73]

1 What is the cube root of 27?
2 (When) you (perform the calculation), subtract 0;7,30 from 27 and
3 you will leave 26;52,30. The 0;7,30 which you subtracted
4 place below 26;52,30 and
5 26;52,30 0;7,30. What is the cube root of 0;7,30? 0;30.
6 Make the reciprocal 0;7,30 and (you get) 8.
7 Multiply 8 by 26;52,30<+>0;7,30 (and you get) 3,36.
8 What is the cube root of 3,36? 6.
9 Multiply 6 by 0;30, the cube root (and you get) 3.
10 3 is the cube root of 27.

which, with decimals substituted for sexagesimal numbers and fractions added, is:

1 What is the cube root of 27?
2 (When) you (perform the calculation), subtract 0.125 (⅛) from 27 and
3 you will leave 26.875. The 0.125 (⅛) which you subtracted
4 place below 26.875 and
5 26.875 0.125 (⅛). What is the cube root of 0.125 (⅛)? 0.5 (½).
6 Make the reciprocal 0.125 (⅛) and (you get) 8.
7 Multiply 8 by 26.875<+>0.125 (and you get) 216.
8 What is the cube root of 216? 6.
9 Multiply 6 by 0.5 (½), the cube root (and you get) 3.
10 3 is the cube root of 27.

The instructions in lines 5 and 7 have caused the most difficulty; but the latter (26.875/0.125 = 215) is followed by 216, *i.e.,* with 1 added to "complete the cube." The procedure uses four inter-related constants applied to 27, 64, 125 and 216 in VAT 8547 to obtain 3, 4, 5 and 6 for the cube roots with (+1) added in Step 3 as follows:

1. The fraction (1/8) = 0.125 to be subtracted from the number whose cube root is to be determined.
2. The integer 8 as a constant multiplier.
3. Unity (1) = 8 • (1/8) to be added (*i.e.*, recombined) to "complete the cube" for which an integer cube root is known.
4. A fixed divisor 0.5 (1/2) is applied to produce the final cube root of the period with the understanding that the cube root of 1/8 is 1/2.

Therefore, for the Ternary, Quaternary, Quinary and Senary "periods" in VAT 8547 of **27, 64, 125** and **216** (years), the cube roots are **3, 4, 5** and **6** obtained in the following consistent manner which (for completeness and to connect with the procedure outlined above) includes the Single [1, **1**, 1, **1**] and Double interval [1, **2**, 4, **8**] given below:

SINGLE INTERVAL (UNITY): [1, **1**, 1 a.u., **1** year]
1. Subtraction of 0.125 from 1 year (1 - 0.125) = 0.875.
2. Multiplication of 0.875 by 8 = 7.
3. Understanding that (7 + 1) = 8, a number with a known integer cube root (2)
4. Reduction of the known integer cube root 2 by 1/2 for the cube root of **1** = **1.**

DOUBLE INTERVAL (BINARY): [1, **2**, 4 a.u., **8** years]
1. Subtraction of 0.125 from 8 years (8 - 0.125) = 7.875.
2. Multiplication of 7.875 by 8 = 63.
3. Understanding that (63 + 1) = 64, a number with a known integer cube root (4)
4. Reduction of the known integer cube root 4 by 1/2 for the cube root of **8** = **2.**

followed by the cube roots for the four numbers in VAT 8547 (**27**, **64**, **125**, **216 years**) :

TRIPLE (TERNARY) INTERVAL: [1, **3,** 9 a.u., **27** years]
1. Subtraction of 0.125 (1/8) from 27 years (27 - 0.125) = 26.875.
2. Multiplication of 26.875 by 8 = 215.
3. Understanding that (215 + 1) = 216, a number with a known integer cube root (6).
4. Reduction of the known integer cube root 6 by 1/2 for the cube root of **27** = **3**.

QUADRUPLE (QUATERNARY) INTERVAL: [1, **4,** 16 a.u., **64** years]
1. Subtraction of 0.125 (1/8) from 64 years (64 - 0.125) = 63.875.
2. Multiplication of 63.875 by 8 = 511.
3. Understanding that (511 + 1) = 512, a number with a known integer cube root (8).
4. Reduction of the known integer cube root 8 by 1/2 for the cube root of **64** = **4.**

QUINTUPLE (QUINARY) INTERVAL: [1, **5,** 25 a.u., **125** years]
1. Subtraction of 0.125 (1/8) from 125 years (125 - 0.125) = 124.875.
2. Multiplication of 124.875 by 8 = 999.
3. Understanding that (999 + 1) = 1000, a number with a known integer cube root (10).
4. Reduction of the known integer cube root 10 by 1/2 for the cube root of **125** = **5**.

SEXTUPLE (SENARY) INTERVAL: [1, **6,** 36 a.u., **216** years]
1. Subtraction of 0.125 from 216 years (216 - 0.125) = 215.875.
2. Multiplication of 215.875 by 8 = 1727.
3. Understanding that (1727 + 1) = 1728, a number with a known integer cube root (12)
4. Reduction of the known integer cube root 12 by 1/2 for the cube root of **216** = **6.**

XVII. "Completing the cube" in astronomical context

Just how this connects with a "Point-Line-Square-Cube" analogy becomes apparent *if* the first number in VAT 8547 is the *Triple Interval's* **27**-year period of revolution and the remaining periods in this text *the sequential Fourth* (**64**), *Fifth* (**125**) and *Sixth Interval* (**216**) in astronomical context.

1 Point^0	*V*i Line^1	*R* Square^2	*T* Cube^3	[Unity, Velocity, Distance, Period] *V*i supplies the names for the Intervals	
1	**3**	9	**27**	Triple	Interval (Ternary).
1	**4**	16	**64**	Quadruple	Interval (Quaternary)
1	**5**	25	**125**	Quintuple	Interval (Quinary)
1	**6**	36	**216**	Sextuple	Interval (Senary)

Table 7a. The Four Intervals in VAT 8547 in astronomical context.

Plus in addition, the inclusion of the Single [1,1,1,1] and Double [1, 2, 4, 8] intervals serves to emphasize that the method extends from unity for all integers, while in passing providing (via the distances) the dimensions 1 x 4 x 9, *i.e.,* those of Sir Arthur C. Clarke's enigmatic, fictional 2001 *monolith* . . .

1 Point^0	*Vi* Line^1	*R* Square^2	*T* Cube^3	[Unity, Velocity, Distance, Period] *Vi* supplies the names for the Intervals	
1	1	**1**	1	Single	Interval, ref. *Vi*, *R*, *T*.
1	2	**4**	8	Double	Interval (Binary)
1	3	**9**	27	Triple	Interval (Ternary)

Table 7b. The Three initial Intervals in astronomical context.

However, an Old Babylonian text (CBS 8165)[74] begins with data for these same three cubes described as follows by Neugebauer and Sachs (1945:34). The missing four cubes (out of seven) from VAT 8547) are suggested on the right.

No. 32. **CBS 1865**. **Fragment of a single table**.
Obverse cube roots beginning with

1e-1 ba-si$_8$
8e-1 ba-si$_8$
27e-1 ba-si$_8$

Only **parts of seven lines** are preserved. The Reverse is inscribed with scattered, half-erased numbers which are obviously connected with the calculation of cubes, e.g., 2,13,20 = (20^3) [emphases added).

No. 32. **CBS 1865**. **single table, 1** through **216**?
Obverse cube roots beginning at 1 with additions to 343, the septenary.

1e-1 ba-si$_8$ Unity
8e-1 ba-si$_8$ Double Interval
27e-1 ba-si$_8$ Triple Interval
64e-1 ba-si$_8$ Quadruple Interval
125e-1 ba-si$_8$ Quinary Interval
216e-1 ba-si$_8$ Senary Interval
343e-1 ba-si$_8$ Septenary Interval

Even so, apart from Earth, with the Double interval uncorrelated, the periods begin with that of the outermost known planet Saturn, and although the last period of 216 years includes the periods of Neptune (163.7232 years) and Uranus (83.7474 years), there is little to connect these values with the next three intervals. However, certain numerical properties for this set are known from later works, including the relation $a^3 + b^3 + c^3 = d^3$. Thus, for the sequence 27 + 64 + 125 = 216, a = 3, b = 4, c = 5 and d = 6, which are the required cube roots of the four Intervals in question. Furthermore, it turns out that *the distances for Uranus* and *Neptune are sequentially inherent in this set* by way of the *geometric means,* initially from the products for the distances, *e.g.*, between the *Fourth* and *Fifth* Intervals (4•5 = 20) for the distance (*R*) of Uranus, and between the *Fifth* and *Sixth* Intervals (5•6 = 30) for the distance (*R*) of Neptune. Both followed in turn by the square roots of the two distances for *Vi* to complete the geometric means with (*T*), the corresponding periods obtainable from Vi^3 or $Vi \cdot Vi^2 = T$, etc,. But either way, note the 30 ("a.u") for the distance of Neptune is the candidate for the "completing the square" exercise and the ellipse for this planet; likewise the 20 (a.u.) for Uranus. and the 9 (a.u.) for Saturn.

1 Point^0	*Vi* Line^1	*R* Square^2	*T* Cube^3	[Unity, Velocity, Distance, Period] *Vi* supplies the names for the Intervals.
1	1	**1**	1	Earth, Single Interval, *T*, *R*, *Vi*, *Vr*
1	$\sqrt{2}$	2	$\sqrt{2 \cdot 2}$	(Gm,) *T* = 2.828427 years
1	2	**4**	8	Double Interval, Binary.
1	$\sqrt{6}$	6	$\sqrt{6 \cdot 6}$	(Gm), *T* = 14.69693 years
1	3	**9**	27	Saturn, Triple Interval, Ternary.
1	$\sqrt{12}$	12	$\sqrt{12 \cdot 12}$	(Gm), *T* = 41.56922 years
1	**4**	16	64	Quadruple Interval, Quaternary.
1	$\sqrt{20}$	**20**	$\sqrt{20 \cdot 20}$	Uranus (Gm),*T* = 89.44272 years
1	**5**	25	125	Quintuple interval, Quinary.
1	$\sqrt{30}$	**30**	$\sqrt{30 \cdot 30}$	Neptune (Gm),*T* = 164.3168 years
1	**6**	36	216	Sextuple Interval, Senary.

Table 7c. The Six integer Intervals & five Geometric means.

Whereas at another time and place the fact that the Double [1,2,4,8] and Triple [1,3,9,27] Intervals predominate in ancient writings includes the condensed arrangement by Proclus in his *Commentary* on the *Timaeus* of Plato: [75]

> "the terms 2, 3, 4, **9, 8** and 27 with reference to the first part, ascribes the principal dignity to the monad."

also noteworthy for the positioning of 9 *before* 8, but understandable with the inclusion of the monad {1} as the standard frame of reference. Thus, although doubled-up, the following familiar arrangements:

1 Point ^{0}	*Vi* Line ^{1}	*R* Square ^{2}	*T* Cube ^{3}
[{**1**} ,	2, 3,	4, **9**,	**8**, 27]

7d. *Proclus*: Combined Intervals.

with the "squaring and cubing" of (*Vi*) = 2 and (*Vi*) = 3 providing the parameters for *both* sets of (*R)* and (*T*).

Lastly, there remains the place of the *Double* interval in all this, for apart from joining with the triple interval as the first integer sets that describe the harmonic law there is no planet *per se* that fits this first interval. Furthermore, the understanding that unity provides the frame of reference is also a necessity for this application. In this respect Macrobius (*ca.*400 CE) in *Commentary on the Dream of Ciprio,* states: "The *monad* represents the point because, like the point, which is not a body but which produces bodies from itself." [76] Emphasis on "bodies" may seem strange, but *astronomical bodies* and their *distances* fits well enough, especially since Macrobius, commencing with 2, next combines all the double and triple numbers to determine "solid bodies." This process includes the cubing of both 2 and 3, followed by the statement that: "In either case the *monad* is necessary to produce a solid body." Thus for the Double, Triple and all such intervals the *monad* (unity) serves as the first parameter and frame of reference, and the second (*Vi*) provides both the *name and the number* to be *squared and cubed* for mean distances and periods as stated earlier. Finally, for expository purposes at least, with the radius (or the heliocentric distance) of Earth 1, and *diameter* of *2*, for the double interval [1, *2*, 4, 8] the corresponding *cube* - length of side *2*, area of one surface 4 (*Vi*^{2}) and volume of 8 (*Vi*^{3}) - are all realized. As, perhaps, similarly reflected in the following assignment by Philolaus:[77]

> The number 8, which the arithmeticians call the first actual cube, has been given by the Pythagorean Philolaus the name of geometrical harmony, because he thinks he recognizes in it all the harmonic relations.
> (Cassidorus, Exp. in Ps., p. 36).

and the following extended summary: [78]

> [According to Pythagoras] . . Number is the first principle, a thing which is undefined, incomprehensible, having in itself all numbers which could reach infinity in amount. And the first principle of numbers is in substance the first Monad, which is a male monad, begetting as a father all other numbers. Secondly, the Dyad is a female number, and the same is called by the arithmeticians even. Thirdly, the Triad is a male number; this the arithmeticians have been wont to call odd. Finally, the Tetrad is a female number, and the same is called even because it is female. . . . . Pythagoras said this sacred Tektractys is: `the spring having the roots of ever-flowing nature.' . . . . the four parts of the Decad, this perfect number, are called number, monad, power and cube.
> (Hippol., Phil,. 2. Dox. 355).

XVIII. The Pythagorean Tetractys and the point-line-area-volume analogy

However, it becomes apparent that the "Double" and "Triple" intervals in Plato's *Timaeus* and the point, line, area, volume analogy are also linked to the Pythagorean *Tetractys*. Which can now be considered in terms of the "lines" and the "areas" in the present "completing the square" exercise. In fact, the first line "I have subtracted the side of my square from the area" leads directly to the groupings of four in the Pythagorean *Tetractys,* beginning simply enough with the first four integers as expanded further in the following commentary by Thomas Taylor: [79]

> The first is that which subsists according to the composition of numbers.
> The second, according to the multiplication of numbers.
> The third subsists according to magnitude.
> The fourth is of the simple bodies.
> The fifth is of figures.
> The sixth is of things rising into existence through the vegetative life.
> The seventh is of communities.
> The eighth is the judicial power.
> The ninth is of the parts of the animal.
> The tenth is of the seasons of the year.
> And the eleventh is of the ages of man.

> All of them however are proportional to each other. For what the monad is in the first and second tetractys, that a point is in the third; fire in the fourth; a pyramid in the fifth; seed in the sixth; man in the seventh; intellect in in the eighth; and so of the rest. Thus, for instance, the first tetractys is 1. 2. 3. 4. The second is the monad, a side, a square, and a cube. The third is a point, a line, a superficies, and a solid. The fourth is fire, air, water, earth. The fifth the pyramid, the octahedron, the icosahedron, and the cube. The sixth, seed, length, breadth and depth. The seventh, man, a house, a street, a city. The eighth, intellect, science, opinion, sense. The ninth, the rational, the irascible, and the epithymetic parts, and the body. The tenth, the spring, summer, autumn, winter. Eleventh, the infant, the lad, the man, and the old man.
>
> The world also, which is composed from these tetractys, is perfect, being elegantly arranged in geometrical, harmonical, and arithmetical proportion; comprehending every power, all the nature of number, every magnitude, and every simple and composite body. But it is perfect, because all things are the parts of it, but it is not itself the part of any thing. Hence, the Pythagoreans are said to have first used the before-mentioned oath, and also the assertion that "all things are assimilated to number."
> Thomas Taylor, *IAMBLICUS' LIFE OF PYTHAGORAS*

As for the formulation of the harmonic law in terms of points, lines, areas and volumes, this is at least systematic in contrast to the routine acceptance of the surprising fact that in the Solar System the cube of a radius vector is equal to the square of the time the latter takes to complete 360° about the "center." Small wonder, then, that one finds included in ancient writings, *e.g.,* Plato's *Epinomis,* the following condensed commentary concerning: [80]

> . . what is called by the very ludicrous name *mensuration*, but is really a manifest assimilation to one another of numbers which are naturally dissimilar, effected by reference to areas. Now to a man who can comprehend this, it will be plain that this is no mere feat of human skill, but a miracle of God's contrivance. Next, numbers raised to the third power and thus presenting an analogy with three-dimensional things. Here again he assimilates the dissimilar by a second science, which those who hit on the discovery have named *stereometry* [the gauging of solids], a device of God's contriving which breeds amazement in those who fix their gaze on it and consider how universal nature molds form and type by the constant revolution of potency and its converse about the double in the various progressions. The first example of this ratio of the double in the advancing number series is that of 1 to 2; double of this is the ratio of their second powers [1 : 4], and double of this again the advance to the solid and tangible, as we proceed from 1 to 8 [1, 2, 2^2, 2^3]; the advance to a mean of the double, that mean which is equidistant from lesser and greater term [the arithmetical], or the other mean [the harmonic] which exceeds the one term and is itself exceeded by the other by the same fraction of the respective terms— these ratios of 3 : 2 and 4 : 3 will be found as means between 6 and 12—why, in the potency of the mean between these terms [6, 12], with its double sense, we have a gift from the blessed choir of the Muses to which mankind owes the boon of the play of consonance and measure, with all they contribute to rhythm and melody. So much, then, for our program as a whole. But to crown it all, we must go on to the generation of things divine, the fairest and most heavenly spectacle God has vouchsafed to the eye of man.
>
> PLATO, *Epinomis* 990c -991b, trans. A. E Taylor (1531-1532)

All of which sheds some light on the emphasis placed on the latter, the triple interval and their relationship to the *Timaeus,* as Macrobius explains at the end of a lengthy passage after introducing the number 7 alone and also in association with 8. The passage concludes by assigning an all-encompassing significance to the number 5 while also providing points of immediate relevance with a hint of more to follow: [81]

> [Chapter V, Macrobius, *Commentary on the Dream of Scipio.*]
>
> [12] Thus it becomes clear that numbers precede surfaces and lines (of which surfaces consist), and in fact come before all physical objects. From lines we progress to numbers, to something more essential, as it were, so that from the various numbers of lines we understand what geometrical figures are being represented.
>
> [13] But we have already remarked that surfaces with their lines are the first incorporeality after the corporeality of bodies and that they are nevertheless not to be separated from bodies on account of their indissoluble union with them. Therefore, whatever precedes surface is purely incorporeal; but we have shown that number is prior to surface and to lines; hence the first perfection of incorporeality is in numbers, and this is, as we previously stated, the common perfection and fullness of all numbers.
>
> [14] The fullness of those numbers which form bodies or bind them together is a particular one, as we suggested above and will explain shortly; at the same time I shall not deny that there are other reasons for numbers being full.
>
> [15] That the number eight produces a solid body has been demonstrated above. But this number has a special right to be called full, for in addition to its producing solid bodies it is also without doubt intimately related to the harmony of the spheres, since the revolving spheres are eight in number. More about this later, however.
>
> [16] All numbers which, when paired, total eight are such that fullness is produced from their union. The number eight is either the sum of the two numbers that are neither begotten nor beget, namely, one and seven, whose qualities will be discussed more fully in their proper place; or from the doubling of that number which is both begotten and begets, namely, four, since four comes from two and produces eight; or from three and five, one of which is the first uneven number of all, while the characteristics of the other will be treated later. [End of Chap. V].
>
> Chapter Vi, Macrobius, *Commentary on the Dream of Scipio.*[82]
>
> [1] FURTHERMORE, *a reason patent to all persuades us that the number seven also deserves to be called full.* But we cannot pass over this fact without first expressing admiration that of the two numbers which, when multiplied with each other, determine the life span of the courageous Scipio, the one is even, the other odd. Indeed, that is truly perfect which is begotten from a union of these numbers. An odd number is called male and an even female; mathematicians, moreover, honor odd numbers with the name Father and even numbers with the name Mother.
>
> [2] Hence *Timaeus,* in Plato's dialogue by the same name, says the God who made the WorldSoul intertwined *odd* and *even* in its make-up: that is, *using the numbers two and three as a basis, he alternated the odd and even numbers from two to eight and from three to twenty-seven.*
>
> [3] The first cubes in either series arise from these: *using the even numbers, two times two, or four, make a surface,* and two times two times two, or eight, make a solid; again, using the odd numbers, thrice three, or nine, make *a surface, and three times three times three, or twenty-seven, the first cube.*

Chapter Vi, Macrobius, *Commentary on the Dream of Scipio.* (continued).

Accordingly we are given to understand that *these two numbers, I mean seven and eight, which combine to make up the life-span of a consummate statesman, have alone been judged suitable for producing the World-Soul for there can be no higher perfection* than the Creator.

[4] This, too, must be kept in mind, that in affirming the dignity belonging to all numbers we showed that they were prior to surfaces, lines, and all bodies; and besides we learned a moment ago that *numbers preceded the World-Soul, being interwoven in it, according to the majestic account in the Timaeus, which understood and expounded Nature herself.*

[5] Hence the fact which wise men have not hesitated to proclaim is true, that *the soul is a number moving itself*. *Now let us see why the number seven deserves to be considered full on its own merits*. That its fullness may be more clearly realized, let us first examine the merits of the numbers whose sums make up seven, then, at last, the capabilities of seven itself.

[6] The number *seven* is made up either of *one and six, two and five, or three and fou*r. *It would be well to treat these combinations separately; we will confess that no other number has such a fruitful variety of powers.*
(Arguments [7] through [18] omitted)

. . . .
. . . .

[19] *The possession of unusual powers came to the number five because it alone embraces all things that are and* seem to be. (We speak of things intelligible as "being," and of things material as "seeming to be," whether they have a *divine or a mortal body*.)" *Consequently, this number designates at once all things in the higher and lower realms*. (emphases supplied)

Macrobius, *Commentary on the Dream of Scipio*, Chapter Vi, Trans by William Harris Stahl. (emphases added)

In point [2] of Chapter Vi, the use of "the numbers two and three as a basis" to construct the "World-Soul" in Plato's *Timaeus* is by now familiar in so much as it incorporates the single line of Proclus (Fig. 4d) :

[2] Hence *Timaeus,* in Plato's dialogue by the same name, says the God who made the World-Soul intertwined *odd* and *even* in its make-up: that is, *using the numbers two and three as a basis, he alternated the odd and even numbers from two to eight and from three to twenty-seven.*

with later statements by Macrobius which refer to seven, a number, which also "deserves to be considered full on its own merits" ([5]).

Further, the number seven is also included as the sum of one and six, two and five, plus (finally) *three and four,* followed by the injunction that "it would be well to treat these combinations separately; we will confess that no other number has such a fruitful variety of powers." ([6]). Grouped in this manner it is difficult *not to* suspect that the Lucas series, *i.e.,* 1,3,4,7,... is part of the dialogue, except, of course, that Francois Lucas (1842-1891) is credited with the (belated) "discovery" of this elementary series.

As for the bald and definite statement in [3] that the numbers seven and eight "combine to make up the life-span of a consumate statesman (and) have alone been judged suitable for producing the World-Soul," this may, perhaps, refer to Plato's *Statesman*. The latter begins with a laborious emphasis on the number three, and later seeks "to define a perfect constitution" from "the rule of one, the rule of the few, and the rule of the many." This is followed by somewhat obscure instructions; "Dividing each of the three into to two let us make six, having first separated the true constitution from all, calling it the seventh," noting next that "Under the rule of one we get kingly rule and tyranny; under the rule of the few, as we said, come the auspicious form of if, aristocracy, and also oligarchy." [83]

It is at this point that the "Number of the Tyrant" in Plato's *Republic* may be considered, with both the latter and the present topic treated in terms of the eighth Pythagorean *tetractys* - "judicial power." Here, according to Macrobius, the number of the (consummate) statesman is explicitly defined as the product of 7 and 8 = 56 years.

This period is readily associated with the second perfect number, 28, or more precisely, twice this value (56). This is an elementary matter, to be sure, but 28 years lies well within the range of periods provided by the apsidal data for Saturn (approximately 27 to 32 years) and also, with 2 periods of revolution for Saturn in 56 years, there will be 3 synodic lap cycles (SD1) and 5 revolutions for Jupiter to complete the (2 : 3 : 5) resonant triple for these two major planets. Furthermore, continuing in the same vein, comprehension is gained from the understanding that the base period of the Pierce planetary framework (and/or its fore-runners) will be 6 times that of the period of Saturn.

Therefore the required base period *P*1 is the product of the first and second perfect numbers (6•28 = 168 years) which can then be applied in standard fashion to the Pierce divisors shown below in Table 5.

Here, however, the correlation with the *Phi*-series begins to deteriorate beyond Jupiter but continues to improve below this planet *en route* to Mercury:

PLANETS N Synodics #	RATIOS (Pierce)	DIVISORS (added)	RES.TRIPLES [(RZT)]	PERIODS (Days) P1/Divisors (JYR)	PERIODS (Years) P1/Divisors (JYR)	*Phi*-series $T = \phi^x$ Exponents:	x
Neptune 1	1/1	1		61,362	168		
Synodic 2-1	1/1	1	1(1)2	61,362	168		
Uranus 2	1/2	2		30,681	84		
Synodic 3-2	1/2	4	1(2)3	15,340.5	42		
Saturn 3	2/3	**6**		10,227	**28**	29.03444185	7
Synodic 4-3	2/3	9	2(3)5	6,818	18.666*	17.94427191	6
Jupiter 4	3/5	15		4,090.8	11.2	11.09016994	5
Synodic 5-4	3/5	25	3(5)8	2,454.48	6.72	6.854101966	4
M-J Gap 5	5/8	40		1,534.05	4.2	4.236067977	3
Synodic 6-5	5/8	64	5(8)13	958.78125	2.625	2.618033989	2
Mars 6	8/13	104		590.01923	1.6153846154	1.618033989	1
Earth/*Syn* 7-6	8/13	169	8(13)21	363.08876	0.9940828402	1.000000000	0
Venus 7	13/21	273		224.76923	0.6153846154	0.618033989	-1
Synodic 8-7	13/21	441	13(21)34	139.14286	0.3809523809	0.381966011	-2
Mercury 8	21/34	714		85.911765	0.2352941176	0.236067978	-3

Table 8. Pierce planetary framework, Solar System & *Phi*-series. Base period from perfect numbers 6•28 = 168 years.

IXX. More on the Triple Interval and mathematical astronomy in Plato's Dialogues

The process of squaring and cubing to determine distances and periods is not only simpler than using fractional exponents, it also lends itself to easy dissemination, especially the integer parameters associated with the triple interval, *e.g.,* [1, 3, 9, 27]: Unity (1), Velocity *V*i (3), Distance R (9) and corresponding period at perihelion (27 years) as applied to Saturn, the outermost planet known in Antiquity. Which in this form plays a role in familiarization as exemplified by the condensed details in Plato's *Republic IX,* 587d–588a.[84] "Three times three, then, by numerical measure," that "by longitudinal mensuration (is) a plane number," followed "by squaring and cubing," after which "it is clear what the interval of this separation becomes." Also, by "taking it the other way about," and if one "tries to express the extent of the interval .... he will find on completion of the multiplication that he lives 729 times as happily and that the tyrant's life is more painful by the same distance." Understandably, these lines represent "An overwhelming and baffling calculation . . . and what is more, it is a true number and pertinent to the lives of men if days and nights and months and years pertain to them." Thus not only the same process, but also, it would seem, the triple interval with both time and distance incorporated in the dialogue, albeit obliquely. Here, commencing with Vi^2 the distance 9 (a.u.) is followed by "squaring and cubing" and the common product 729 ($T^2 = 27^2$ and $R^3 = 9^3$) included for good measure. On the other hand, the confusion caused by references to: "life," "the tyrant" and "pain and pleasure" in the above become somewhat more understandable when the methodology inherent in the Pythagorean *Tetractys* is taken into consideration.

But above all else, it is the suitability of the Triple interval [1, 3, 9, 27] - Ref. Unity, velocity *V*i = 3, Distance 9 (a.u) and corresponding period of revolution of 27 years applied to Saturn which is the most helpful. Then again, the data pertain to the shortest period (27 years) associated with perihelion, and not the mean period or longest period at aphelion, even though the latter was the result of the "completing the square" approach to elliptical orbits. Nor for that matter does rounding the aphelion period to 32 years help greatly either since the distance 10 (a.u.) now becomes 10.079368 (a.u.). And also, multiplication of the velocity (3.162277) by 10, the corresponding distance to produce the period *T* of 31.622777 years is also lost. Although not lost entirely, since in condensed form it is again relation (4) from the 1989 Galileo paper discussed earlier, *i.e.,* $T = Vi^3$.

APSIDES	Unity	Velocity Vi	Distance R	Period T
Perihelion	**1**	**3**	**9**	**27**
SATURN	1	3.0822070	9.5	29.280967
Aphelion	1	3.1622777	10	31.622777
Perihelion	**1**	**3**	**9**	**27**
SATURN 2	1	3.0898733	9.547317	29.5
Aphelion	1	3.1748021	10.079368	32

Table 9. The Triple Interval and integer values

Whereas the 32-year aphelion period assigned to Saturn reappears among the workings of the "Alchemists" in the far from straightforward context supplied below. Although still a difficult text, towards the end some degree familiarity with material presented here in the preceding pages becomes apparent. In particular, the question:

"If the Capacity or Circumference of the sphere be 32 foot, how much will one of the sides of the Cube be to Equalize the Capacity of this Sphere?"

at least begins to swing the dialogue towards the "Point-Line-Area-Volume" analogy, "completing the cube," and a reference to Saturn, plus "six the first of the perfect numbers."

Atalanta fugiens emblems 21 - 25

Michael Maier's alchemical emblem book *Atalanta fugiens* was first published in Latin in 1617. It was a most amazing book as it incorporated 50 emblems with epigrams and a discourse, but extended the concept of an emblem book by incorporating 50 pieces of music the 'fugues' or canons. In this sense it was an early example of multimedia.[85]

Emblem XXI. [transcribed by Hereward Tilton]
Make of the man and woman a Circle, of that a Quadrangle, of this a Triangle, of the same a Circle and you will have the Stone of the Philosophers.
. . . . .
[Closing paragraphs]
But that this was not unknown to the Philosophers of Nature is apparent from this: That they command a Circle to be turned into a Quadrangle, and this by a Triangle to be reduced again to a Circle. By a circle they understand the most simple body without angles, as by the Quadrangle they do the four Elements. It is as if they should say: The most simple corporeal Figure that can be found is to be taken and divided into four Elementall Colours, becoming an Equilaterall Quadrangle. Now every man understands that this Quadration is Physicall and agreeable to Nature, by which far more benefit accrues to the Publick, and more light appears to the mind of Man, than by any meere Theory of Mathematicks when abstracted from Matter. To learn this perfectly a Geometrician acting upon solid bodyes must enquire what is the depth of solid Figures, as for example the Profundity of Sphere and Cube must be knowne and transferred to manuall use and practice. If the Capacity or Circumference of the sphere be 32 foot, how much will one of the sides of the Cube be to Equalize the Capacity of this Sphere? On the contrary, one might look back from the Measures which the Cube contains to the feet of each Circumference.
In like manner the Philosophers would have the Quadrangle reduced into a Triangle, that is, into a Body, Spirit and Soul, which three appear in the three previous colours before Rednesse: that is, the Body or earth in the Blacknesse of Saturn, the Spirit in the Lunar whitenesse as water, and the Soul or air in the Solar Citrinity. Then the Triangle will be perfect, but this again must be changed into a Circle; that is, into an invariable rednesse, by which operation the woman is converted into the man and made one with him, and six the first of the perfect numbers is absolved by one, two having returned again to an unity in which there is Rest and eternall peace.

Closing paragraphs for *Atalanta fugiens* emblem 21 by Michael Maier (1617). Introduced by Adam Mclean.[85]

The cryptic description of Emblem XXI "*Make of the man and woman a Circle, of that a Quadrangle, of this a Triangle, of the same a Circle and you will have the Stone of the Philosophers*" is discussed in Part Four with respect to isosceles and equi-lateral triangles emphasized in Plato's *Timaeus.* Linked to the "Rotation of the Elements," it is shown that the former pertain to the Fibonacci series with ratios between the sides of the triangles increasing towards *Phi* itself as the rotations continue. The Lucas series variant makes use of the same rotations but differs in range and purpose as it continues to approximate the ½*Phi*-series, improving towards this outcome with each successive rotation.

XX. The Body, Spirit and Soul Triad

But there is also something else of importance, which is the reference to "a triangle" reduced into "a Body, Spirit and Soul." Of the three, "body" is perhaps the simplest since it is a well attested term for an astronomical body. Furthermore, with this key providing the way, there are sufficient examples in the literature to also assign "spirit" to Velocities (*V*i & *V*r) and "soul" to Time (T,S), but not without some difficulty in the absence of additional information.

In terms of the extant alchemical literature, Philip à Gabella (1615) perhaps leads the way by passing on that: [86]

according to the first fathers of philosophy the magical contemplation of t**he ternary** encompassed **body, spirit** and **soul**

while in more detail, Johann Isaac Hollandus in *Of natural & supernatural things*. London,1670, writes: [87]

They say, in our Stone are the four Elements, and they say true; for the four Elements must be separated out of **Saturn**. They say, our Stone consists of **Soul, Spirit** and **Body**, and these three become one. They say true; when it is made fixed for the white **Mercury** and Sulphur with its' Earth, then these three are one.

and in *Paracelsus his Aurora, & Treasure of the Philosophers* (1659) where "Mercurius" states in part: [88]

"This mystery it is permitted only to the prophets of God to know. Hence it comes to pass that this Stone is called animal, because in its blood a **soul** lies hid. It is likewise composed of **body, spirit, and soul.** For the same reason they called it their microcosm, because it has the likeness of all things in the world, and thence they termed it animal, as Plato named the great world an animal."

plus more from *The naturall Chymicall Symboll* or *short Confession of Doctor Kunwrath*: [89]

[**Three in One, One in Three**]. But this is the true philosophicall doctrine of the philosophers **Mercury**, That Three is One generall Chaos, Three in essence, namely **Body, Soule**, and **Spiritt**; and these Three Essences are had in One substance or thing and neere at hand.

[**Body Soule Spirit**.] There is one Essence of the **Body** one other of ye **Spirit**, one other of the **Soule; **But ye **Body, Soule,** and **Spirit** are one thing, wherein all the three are together equally necessarily present at the same time.

For the Body is not made by the Arte of Man, nor is the **Spirit** made by the Arte of Man, neither is the **Soule** made by the Arte of Man. (all emphases supplied)

Macrobius, on the other hand, in his *Commentary on the Dream of Scipio* is more direct while at the same time invoking the Double and Triple intervals with emphasis on assigning the last exponent of the Point-Line-square cube analogy to a solid body, then finally arriving at Soul = time and motion "with its animating power" and non-corporeal form: [90]

[12] Since the monad is the source of even and uneven numbers alike, the number three should be considered the first line. This tripled gives nine, which from its two lines, as it were, produces a **body** with length and breadth, as was the case with the number four, the second of the even numbers. In the same way the number nine tripled supplies the third dimension. Thus with the uneven numbers a solid **body** is formed in twenty-seven, three times three times three, just as with even numbers eight or two times two times two made **a solid body**.

[13] In either case **the monad** is necessary to produce **a solid body**, in addition to the other six numbers, three even and three uneven, two, four, and eight being the even numbers and three, nine, and twenty-seven the uneven.

[14] Plato's *Timaeus*, in disclosing the divine plan in the creation of the World-Soul, said that **the Soul** was interwoven with those numbers, odd or even, **which produce the cube or solid,** not meaning by this that the **Soul** was at all corporeal; rather, in order to be able to penetrate the whole world with its animating power and fill the **solid body** of the universe, the **Soul** was constructed from the numbers denoting solidity. (all emphases supplied)

A small selection, to be sure and also not particularly compelling, but the complexity of the matter can at least be given a detailed basis in the form of the *Phi*-series planetary framework generated earlier in Table 3. As shown below, a reduced, rearranged version of the latter has double or triple occurrences of parameters in each of the three categories - Soul (Periods *T*), Body (Distances *R*) and Spirit (Velocity *Vi*) with (*Vr*) not part of the set but available:

PLANETS N Synodics #	Unity Point, Vi^0	SPIRIT *(Vi)* Line, Vi^1	BODY *(R)* Square, Vi^2	SOUL *(T)* Cube, Vi^3	$= \phi^x$ x	Velocity *(Vr)* (Reciprocal Vi)
Saturn 3	1	3.073532624	9.446602789	29.03444185	7	0.325358512
Synodic 4-3	1	2.618033989	6.854101966	17.94427191	6	0.381966011
Jupiter 4	1	2.230040414	4.973080251	11.09016994	5	0.448422366
Synodic 5-4	1	1.899547627	3.608281187	6.854101966	4	0.526441130
M-J Gap 5	1	1.618033989	2.618033989	4.236067977	3	0.618033989
Synodic 6-5	1	1.378240772	1.899547627	2.618033989	2	0.725562630
Mars 6	1	1.173984997	1.378240772	1.618033989	1	0.851799642
Earth/*Syn* 7-6	1	1.000000000	1.000000000	1.000000000	0	1.000000000
Venus 7	1	0.851799642	0.725562630	0.618033989	-1	1.173984997
Synodic 8-7	1	0.725562630	0.526441130	0.381966011	-2	1.378240772
Mercury 8	1	0.618033989	0.381966011	0.236067978	-3	1.618033989

Table 3s. *Phi*-series *Vr*, *Vi*, *R*, *T* and the *Point-Line-Square-Cube / Spirit-Body-Soul* assignments.

XXI. The World-Machine of Francesgo Giorgi (1525).

The significance of certain sets of data is more apparent for some than others; for example, the primary numbers for the Synodic 6-5 position are, from right to left the three constants assigned to the mean planetary Periods, *i.e*., relation (8), the Distances of the Planets, relation (9) and the Periods for the planet-synodic expansions, relation (12). Other assignments of interest include the three locations for *Phi* itself, firstly as the Period (SOUL) of Mars, secondly as the Velocity *Vr* of Mercury and thirdly also the Velocity (*Vi*) (SPIRIT) for the Mars-Jupiter Gap. Whereas the three occurrences for the synodic 8-7 period *T* (SOUL) between Venus and Mercury are the Distance *R (BODY)* for Mercury, and again the Velocity (*Vr*) for synodic4-3 between Jupiter and Saturn. Which, in addition to dynamic aspects, also involves the first of unities" ("384") in a summation concerning the construction of a "world-machine" by Francesco Giorgi (1466-1540) whose *Harmonia Mundi* (1525) concludes with a remarkably condensed statement apparently concerned with the construction of a "world-machine" as opposed to the "world-soul" of Plato: [91]

Thus this world-machine, consisting of a simple number squared and cubed, leads to a sovereign concord, not only of the ternary to 27 but of the senary to 162 and from 384 to 10,368, as we have set them out above.
Francesco Giorgi, *Harmonia Mundi* (Venice, 1525)

The statement is, however, readily decoded, especially if the reader is familiar with both the "Ternary" and "Senary" (the Triple interval [1, 3, 9, 27] and the Sextuple interval [1, 6, 36, 216] respectively) in the present context. This said, it appears that there are two significant activities involved, firstly with the *Phi*-series concerning the number "27" and the range "from 384 to 10,368" which concerns heliocentric planetary distances. And secondly, a separate approach which involves providing the upper limit for the outermost planet ("Neptune") of 162 (years) as given.

Remaining with the latter there is therefore now a further base period for the Pierce planetary framework and its divisors. Although the resulting period for Mercury is low (162/714 = 0.227209 years) the division by 6 for Saturn, the outermost *known* planet in Antiquity (162/6) results in 27 years, which is also the period/volume parameter (or Soul again in years) of the Ternary. Whereas the product: 27•384 = 10,368, the range assigned in the text. Except for one thing, which is to provide a meaning for this data the range is best understood to extend from 0.384 "a.u." to 10.368 "a.u." in 27 steps. As for the hows and the whys of this matter, both are influenced by an earlier 36-step sequence, which is fortunately explained in the meticulous footnotes accompanying another relatively obscure work, this time by George Burges entitled: *THE TREATISE OF TIMEUS THE LOCRIAN ON THE SOUL OF THE WORLD AND NATURE* (London,1876).

Here Burges explains the relationship (and more) between the 36-step and 27-step configurations augmented by recognizable references to the Binary [1,2,4,8], Ternary [1,3,9,27] and the point-line-square-cube analogy. Plus something else, which is an extension to incorporate the "Soul of the World." [92]

> … But why were the terms fixed at 36 ? The reason is to be found in the mysteries of the school of Pythagoras, where it was thought proper to multiply 384, the assumed term. By 27. But why 27? Because that number is the sum of the first numbers, which must represent lines, surfaces, solids, squares, and cubes, added to unity. Thus 1 is unity ; 2 and 3, the first numbers representing lines : 4 and 9, the first surfaces, and both squares, the former of an even number (2), and the latter of an odd number (3). Taking then the number 27 as the symbol of the world, and the numbers which it contains as the symbols of the elements and their combinations, it was reasonable for the Soul of the world, which is the very basis of order and of combinations, which constitute the world, to be composed of the same elements (of order) as the number 27 in itself.
>
> George Burges, Supplement to *TIMEUS THE LOCRIAN* (1876 : 172-173)

This leaves the matter of the "First of Unities" - ("384") also discussed by Burges in *TIMEUS THE LOCRIAN*, *The Works of Plato : A New Literal Version* Vol. VI. (1876: 150-151), which in a technical, dynamic sense turns out to be far from simple. It is also, as a variation of other interpretations of Plato's *Timaeus*, impressive in the manner by which the "Body" = Distance (Square), and "Soul" = Period (Cube) methodology discussed earlier is extended and linked to the "First of Unities." [93]

> But the soul of the world has (the deity) united with the centre and led it outwards, investing the world wholly with it, and making it a mixture of Form undivided, and of Substance divided, so as to become one mixture from these two; for which (world) he mixed up two threes, the origin of motion, one connected with the same, the other with the different; which (soul), being mixed with difficulty, was mixed not in the easiest way. Now all these proportions are combined harmonically according to numbers: which proportions he has divided according to a scale scientifically, so that a person is not ignorant of what things and by what means the soul is combined; which the deity has not ranked after the substance of the body,– for, as we say, that which is before is in greater honor as regards both power and time, – but he made it older by taking the first of unities, which is 384.
>
> George Burges, *TIMEUS THE LOCRIAN* (1876 : 150-151)

To understand the First of Unities it is sufficient to know that for the *Phi*-series the "Body," or heliocentric distance of Mercury is equal to the "SOUL," or period of the Mercury lap-cycle with respect to Venus, *i.e.*, Synodic 8-7. Moreover,

PLANETS N Synodics		ϕ^x = Series	SOUL *(T)* Period/Time	BODY (*R*) Distance (a.u.)	SPIRIT (*Vi*) Velocity	Reference Unity
NAMES	#	x	Cube, Vi^3	Square, Vi^2	Line, Vi^1	Unity Vi^0
Venus	7	-1	0.618033989	0.725562630	0.851799642	1
Synodic	8-7	-2	0.381966011	0.526441130	0.725562630	1
Mercury	8	-3	0.236067978	0.381966011	0.618033989	1

Table 3tr. Synodic position 8-7. *Phi*-series *T*, *S and Vi*, Mercury through Venus.

in the same wise, this number is therefore 0.384 in both applications, *i.e.*, 0.384 a.u. for the distance and 0.384 years for the period. Which then results in the range from 0.384 to 10.368, with the latter the aphelion distance of Saturn as opposed to the lesser perihelion and mean distances. But even so, the increment for the 27 steps obtained from (10.368 - 0.384)/27 = 9.984/27 is 0.369777* with the sequence ending at 33.384316 years for the distance (BODY, R) of 10.368 a.u. at the 27th step.

The interplay between "Body" and "Soul" by Burges in this passage plus the reference to "power and time" and the odd statement that the Creator "made it older by taking the first of unities, which is 384," is one method of placing the second instance of the parameter which occurs as the Distance of Mercury in the location of a second category. Thus also now associated with Time (Soul) and the "older" Mercury-Venus lap-cycle, Synodic 8-7 while utilising the the pheidian exponents -1, -2, -3 for the period (*T,S*) for Venus (*T*), Mercury-Venus Synodic 8-7 (*S*) and Mercury (*T*).

The reason for this odd duality - at least from a bare numerical viewpoint - is straightforward enough, simply the consequence of the *PhI*-series applied sequentially downwards to produce the periods (*T,S*) and then using Kepler's Third Law of planetary motion to determine the Distance of Mercury. Thus, for the *Phi*-Series, ϕ^x, x = -1 -2, -3 , . . n) first the period of revolution of Venus = ϕ^{-1}, next the period for Synodic 8-7 = ϕ^{-2} and last, the period of Mercury = ϕ^{-3}. However, in exponential terms, applying the conversion from Period to Distance, *i.e*., $R = T^{2/3}$ the latter applied to Mercury ϕ^{-3} results in $(\phi^{-3})^{2/3} = \phi^{-2}$, which is identical to the exponent which generates Synodic 8-7 and the duality.

Remarks

Just when and how this situation came to be uncovered is difficult to say, of course. Perhaps it was Pythagorean, perhaps earlier, or perhaps a little later. But in any event, there are increasingly more questions than answers at the present time, and it is likely that there are more aspects buried in the *Tetractys* which may have roles to play. For example, further divisions arising from Odd and Even numbers and their Male and Female assignments, the social, legal and political aspects, and the complexities embraced by factoring in "Life" and "Nature" itself.

Certainly among the alchemists various bits and pieces are discernable; witness now a partial agreement with the last remark above by Marcilio Ficino (1518), who states: [94]

> For although sulphur and Mercury were as it were the root of metals before the first coagulation, yet now they are not, since they are brought to another nature: whence it remains that there cannot be made out of them any metallic body. Since also the chain is unknown, by which Venus & Mercury copulated together in due proportion.
>
> Marsilius Ficinus, 'Liber de Arte Chemica', in *Theatrum Chemicum,* Vol 2, 1702.

thus a "hermaphroditic" component associated with the adjacent positions for Synodic 8-7 and Mercury a further complication. Here, however, in an extract from Bernard of Trevisan, *Le Texte d'Alchymie et le Songe-Verd,* (1695) a slightly more understandable account is offered: [95]

> Therefore, my child, you see very well that I have declared all to you when I have made you understand in what manner our Sulfur is contained in the belly of the Mercury, and that it is correct to call it internal Sulfur or hidden Spirit, which is no other thing than heat and dryness, acting on the cold and the moisture, acting on the patient, the pure mercurial substance of which Sulfur is the Soul, since it is it which vivifies and sustains the Mercury which would be, without our Sulfur, only a dead, unfruitful, and sterile earth. There is then good reason to say that Sulfur and Mercury are the proper and true substances of the metals, because it is very certain that this Sulfur cannot be without Mercury and that our Mercury cannot be without this Sulfur which is intimately united and incorporated with it, as the soul is with the body.
>
> These two names of Mercury and of Sulfur are only names for one single substance which we know under the names of Quicksilver or Mercury.
>
> Therefore make manifest that which is hidden and make occult that which is manifest. I tell you, in that alone consists the work of the sages. Our gum curdles our milk, and our milk dissolves our gum, and they grow in the Stone of Paradise, which Stone is of two contrary natures, that is to say, of the natures of Fire and of Water.
>
> All that I have written above ought to have opened your understanding to the intelligence of the philosophers - for what I have explained to you altogether well and have given you to understand what our Sulphur is, that the philosophers have also called Gum, Oil, Sun, Fixity, Red Stone, Curd, Safran, Poppy, Red Brass, Tincture, Dry, Fire, Spirit, Agent, Soul, Blood, Burned Brass, Red Man, and Quick Earth. I have also given you a clear and concise explanation of that which the philosophers name Water, Milk, White Wrapper, White Manna, White Urine, Cold, Moisture which does not dampen, Body, Womb, Moon, White Woman, Changing Habit, volatile, patient, Virginal Milk, Lead, Glass, White Flower, Flower of Salt, Fleece, Veil, Venom, Alum, Vitriol, Air, Wind, Rainbow, Naked Woman, and so many other names which are only for the purpose of making us conceive the qualities, properties, and the two natures of male and female contained in our substance, which is nothing else but animated Quicksilver. It is this viscous moisture mixed with its earthy part, our Mercury, and the true foundation of all our science.
>
> It is in this great number of terms that the wise men have taken pleasure in writing their sentiment relative to our science. All these names ought to convince you of the truth of our science, for all of them have only one meaning and all of them have for their purpose only to expose the hermaphroditic Mercury to us. It is feminine if it is considered as separated from the Sulfur which it contains within it and of which it is the substance; but it is masculine if it is considered according to its Sulfur with which it is united so intimately that it cannot be separated from it; and it can be said of their marriage that they are both of them in the same flesh.
>
> Bernard of Trevisan, *Le Texte d'Alchymie et le Songe-Verd* (1695).

But there is something else which intrudes, for in addition to the mention of Mercury and sulphur (the latter called here the soul) is what can be understood to be the third occurrence of 0.384 as an "internal sulphur or hidden spirit," the latter part of which is undoubtedly correct, since:

Distance (Body) of Mercury = Mercury-Venus Synodic (Soul) = Velocity (Spirit) of Jupiter-Saturn Synodic 4-3.

Moreover, in all three instances and in three separate assignments the parameter for the distance, the period and the velocity is - for the *Phi*-series - the number 0.381966011. Which, if rounded at the third decimal place, becomes 0.382 and by multiplying by 1,000 the convenient integer 382. But wither and whence came the "First of Unities"? And was it something more, or possibly an alternate event after noting Marcilio Ficino's observation that "the chain is unknown, by which Venus and Mercury copulated together in due proportion" ? More intriguing is the apparent linkage between the two massive superior planets Jupiter and Saturn with the two innermost and the smallest, with Earth, perhaps errantly, somewhere in between.

On the other hand, after the positive results obtained from the mean value Pheidian and Fibonacci ratios inherent in the motions of these two gas giants, and knowing that for the *Phi-series* that the inverse velocity for Synodic 4-3 between the two is ϕ^2 with its reciprocal the relative velocity Vr, one might start by using the Fibonacci ratio 13/5 = 2.6 versus 2.61803398 …and take the reciprocal, or simpler yet, use the Fibonacci ratio 5/13 = 0.38461538. . . Or again use 2.604 to similarly obtain 0.384024 … which readily rounds to 0.384. It is better, however, to concentrate on the relative motions of Jupiter with respect to Saturn which already have some peculiarities in the case of the data for the former. These include an unexpected difference between the ranges for Babylonian Systems A and B for Jupiter with nothing comparable known for Saturn. How might this be linked to the First of Unities? Simply by investigating the relative motions and the two sets of data for Jupiter and Saturn to this end, but only after checks using modern data to provide a standard for their historical equivalents.

Thus for the modern Solar System, using 29.42351935 years (T_1) and 11.85652502 years (T_3) for the mean periods of revolution and applying *Phi*-series/synodic relation (1), the synodic period or lap cycle of Jupiter with respect to Saturn (S_2) is 19.85887209 years with a velocity (Vr) of 0.36927… *versus* the target value of 0.384. Similarly, for the Babylonian mean periods (11.86111* and 29.444* years) the synodic period (S_2) is 19.86220818 and the velocity is 0.36925, thus largely unhelpful results. But this is far from the end of the matter, especially in terms of the *Phi*-series and Babylonian methodology.

First of all, the data for the periods of revolution and the intervening synodic cycles generated by the *Phi*-series - (ϕ^x, x = -3 -2, -1, 0, 1 , 2. . 7) involve the use of *odd* and *even* exponents for the planets and synodic cycles respectively with exponents -3 and -2 for the periods of Mercury and the synodic cycle between the latter and Venus, whereas exponents 5, 6 and 7 are used for Jupiter, Synodic 4-3 and Saturn. For the Synodic 4-3 data the period is therefore ϕ^6 with the relative velocity *Vr* obtained from $(\phi^6)^{-1/3} = \phi^{-2} =$ 0.381966011and the Distance *R* for Mercury derived from $(\phi^{-3})^{2/3} = \phi^{-2}$. Lastly, the period of Synodic 8-7 remains ϕ^{-2} as generated by the *Ph*i-series and incorporated in Tables 3 and 3s. Plus, of course, a more accurate though lower value for the First of Unities of 382.

Secondly, as already seen, the calculation of the mean synodic arc in Babylonian astronomy involves a further period other than the mean periods of revolution and synodic difference periods, namely the parameter *P* which involves both the period of revolution $P = (T - 1)$ and $P = 360/(u)$, the number of mean synodic arcs per revolution of 360°. Therefore, for Jupiter P_J is 10.86111* years and for Saturn P_S is 28.444* years with the synodic period now 17.56994909 and the corresponding velocity 0.38465…Furthermore, the same procedure applied to the modern periods likewise results in a velocity of 0.38468 . . . thus purists would likely round to 0.385, but from a practical viewpoint 0.384 might nevertheless be preferred in both instances.

This is still not quite the end of this "short" excursus, because the two new synodic periods are both close to the *Phi*-series' ϕ^6 (17.94427191) *versus* 17.56994909 and 17.56593320, with 1.612359016 and 1.6122975888 from the sixth roots of the new periods versus *Phi* itself. These results and others like them helped to provide the impetus for the real-time investigations of planetary motion for these two planets in particular discussed further in Part III.

So much, then for the First of Unities and the minor incursions into the complexities of past and present Alchemy leaving only one thing remaining. Which is the issue arising from the data from Aaboe's 1964 limited research into a possible Babylonian System B augmented here by Friberg's approach to parameters in the ratios of 80/81 and 81/80, thus reciprocals and in the format adopted for the Babylonian velocities, M = 1;00,45° and u = 0;59,15,33,20°. To which can be added (in terms of symmetry) the minimum arc m = 0;57,46,6,40°.

Next after the conversion from arcs to time and again to distance relative to unity, the ratio of the distances $(M_d - m_d)/(M_d + m_d)$, *i.e.*, Green's method[96] for the calculation of eccentricities, yields 0.01677279 *versus* 0.016708617 from modern estimates.[97]

Bibliography and References

EXCURSUS

28. Neugebauer, Otto. *Astronomical Cuneiform Texts,* Planetary Theory, dual form period relations. ACT (1955:282).
29. Sachs, Abraham in *Astronomical Cuneiform Texts,* Jupiter synodic arc extent n ACT 817, Section 2, line 8. ACT (1955:430).
30. ______________ in *Astronomical Cuneiform Texts,* Jupiter period of invisibility re ACT 817 Section 2. ACT (1955:430).
31.______________in *Astronomical Cuneiform Texts,* Jupiter 427-year period relation in ACT 813, Section 20 & 21. ACT (1955:404).
32. Harris, John N. *Letter to the Editor of ISIS* re determination of Babylonian period relations. *ISIS*,
33. Neugebauer, Otto. Trans. *Astronomical Cuneiform Texts,* deriv., Jupiter mean synodic arc. ACT 813, Section 21. ACT (1955:414).
34. Neugebauer, Otto. Trans. *Astronomical Cuneiform Texts,* deriv., Saturn mean synodic arc of Saturn. ACT 802, ACT (1955:373).
35. ________________ Trans. *Astronomical Cuneiform Texts,* possible derivation System B for Mars. ACT 811a, II. ACT (1955:390).
36. Neugebauer, Otto. *Astronomical Cuneiform Texts,* 1151-year period and Theory of Venus, ACT (1955:300).
37. ________________ *Astronomical Cuneiform Texts,* 46-year period and Theory of Mercury, ACT (1955:287).
38. ________________ *Astronomical Cuneiform Texts,* Four characteristic phenomena periods for Mercury, ACT (1955:288).
39. Neugebauer, Otto. Neugebauer, Otto. *Astronomical Cuneiform Texts: Babylonian Ephemerides of the Seleucid Period for the motion of the Sun, The Moon, and the Planets.* 3 vols, Lund Humphreys, London, 1955.
40. Neugebauer, Otto. *A History of Mathematical Astronomy,* Springer-Verlag, New York, 1975.
41. van der Waerden, Bartel. *Science Awakening II, The Birth of Astronomy,* Nordhoff Intl. Leyden & Oxford U. Press, NY 1974.
42. Swerdlow, Noel. *The Babylonian Theory of the Planets,* Princeton University Press, Princeton, 1998.
43. Neugebauer, Otto. *A History of Mathematical Astronomy,* Planetary Theory, HAMA (1975:347-348).
44. ________________ *A History of Mathematical Astronomy,* Planetary Theory, HAMA (1975:1113).
45. Newton Robert R. *The Crime of Claudius Ptolemy,*
46. ________________ *A History of Mathematical Astronomy,* Checking Ptolemy's velocities. HAMA (1975:???).
47. ________________ *The Origins of Ptolemy's Astronomical Parameters*, Technical Publication No. 4, A cooperative publication by The Center for Archaeoastronomy, University of Maryland, and The Johns Hopkins University, applied Physics Laboratory, Maryland, 1982.
48. Bernstein, Bernard R. A*l-Bitruji: On the Principles of Astronomy,* Yale University Press, New Haven, 1971.
49. Duncan, A. M. *Copernicus: On the Revolutions of the Heavenly Spheres*, (trans.). Barnes and Noble, New York 1976.
50. Manitius, K. *PTOLEMAUS, Hanbuch der Astronomie*, Band II, B.G. Teubner, Leipzig, 1963.
51. Pedersen, Olaf, *A Survey of the Almagest,* Odense University Press, Odense, Denmark, 1974.
52. Neugebauer, Otto. *A History of Ancient Mathematical Astronomy,* HAMA 1975:279).
53. _________________ *Astronomical Cuneiform Texts,* Planetary Theory in General, ACT(1955:279).
54. ________________ *Astronomical Cuneiform Texts,* Leading Ideas of Planetary Theory. ACT(1955:279).
55. ________________ *Astronomical Cuneiform Texts*, Characteristic Phenomena, ACT (1955: 279).
56. ________________ *Astronomical Cuneiform Texts*, Babylonian "natural approach." ACT (1955: 280).
57. ________________ *Astronomical Cuneiform Texts*, Jupiter synodic time in ACT 812, Sect 1 13;30,27,46 MSM. ACT (1955: 280).
58. ________________ *Astronomical Cuneiform Texts*, Transcription & Commentary I on ACT 812, Section 1. ACT (1955: 280).
59. ________________ *Astronomical Cuneiform Texts*, Transcription & Commentary II on ACT 812, Section 1. ACT (1955: 393).
60. ________________ *Astronomical Cuneiform Texts*, Transcription & Commentary II on ACT 813, Sec 14,15,16. ACT (1955: 412).
61. ________________ *Astronomical Cuneiform Texts*, Transcription & Commentary II on ACT 814, Section 2. ACT (1955: 424).
62. Emiliani, Cesare. *The Scientific Companion,* John Wiley & Sons, New York, 1995:121.
63. Green, Robin S. Green, R. M. *Spherical Astronomy*, Cambridge University Press, Cambridge 1985:141.
64. du Sautoy, Marcus. Open University, http://www.open2.net/historyandthearts/discover_science/mathsproblems.html.
65 Moore, Patrick. *Naked Eye Astronomy*, W. W. Norton, New York, 1965:177.
66. Webb, T. W. *Celestial Objects for Common Telescopes,* Dover, New York, 1962:221.
67. Levy, David H. *THE SKY - A User's Guide*, Cambridge University Press, Cambridge, 1991:134.
68. Morgan, Morris Hicky. *VITRUVIUS: The Ten Books on Architecture*, Bk III, Ch. III,10. Dover Books, New York 1960:84
69. Harris, John N. "Velocity Expansions of the Laws of Planetary Motion," *Journal of the Royal Astronomical Society of Canada*, Vol. 83, No. 3, pp. 207-218, June 1989.
70. Galilei Galileo, *Dialogues Concerning Two New Sciences* (1638), Fourth Day, translated by Henry Crew and Alfonso de Salvio (1914). Dover Publications, Inc., New York, 1954.
71. Neugebauer Otto, and A. Sachs, CBS 8156, *MATHEMATICAL CUNEIFORM TEXTS ,* (MCT), New Haven, 1945: 34
72. Sachs, Abraham, "Babylonian Mathematical Texts II-III," *JCS* Vol. 6, No, 4, 1952:151-155
73. Muroi, Kazuo. "Extraction of Cube Roots in Babylonian Mathematics," *CENTAURUS*, Vol. 31, 1989:181-188.
74. Neugebauer, Otto and Abraham Sachs. Translation and Commentary on CBCS1865 in *Mathematical Cuneiform Texts*,
75. Taylor, Thomas. *The Commentary of Proclus on the Timaeus of Plato*, (5 Vols). Kessinger Publications, Kila, Montana, USA. ISBN 1-56459-349-5, Vol. 1, pp. 1-11
76. Stahl, William Harris. *Macrobius: Commentary on the Dream of Scipio*, Columbia University Press, New York, 1952: 97.
77. Cassidorus, Exp. in Ps., p. 36, in *THE PYTHAGOREAN SOURCE BOOK AND LIBRARY,* Translation & compilation by Kenneth Sylvan Guthrie (orig. 1920). Phanes Press, Grand Rapids, 1988:169.

Bibliography and References

EXCURSUS (cont.)

78. Hippol., Phil,. 2. Dox. 355, in *THE PYTHAGOREAN SOURCE BOOK AND LIBRARY,* Translation and compilation by Kenneth Sylvan Guthrie (orig. 1920). Phanes Press, Grand Rapids, 1988:212
79. Taylor, Thomas. *IAMBLICUS' LIFE OF PYTHAGORAS*, Inner Traditions International, Ltd. Rochester 1986:235.
80. *Epinomis* 990-992e, Trans. A. E. Taylor in *PLATO: The Collected Dialogue*s, Edited by Edith Hamilton & Huntington Cairns, Princeton University Press, Princeton, 1961:1531-1533.
81. Stahl, William Harris. *Macrobius: Commentary on the Dream of Scipio*, Columbia University Press, New York, 1952: 97.
82. __________________ *Macrobius: Commentary on the Dream of Scipio*, Columbia University Press, New York, 1952: 190.
83. Plato. *Statesman,* 302c-303b. Trans. J. B. Skemp in *PLATO: The Collected Dialogues of Plato*, Princeton University Press, Princeton, 1982: 1073-1074.
84. Plato. *Republic* IX, 587d–588a. Trans. Paul Shorey in *PLATO: The Collected Dialogues of Plato*, Princeton University Press, Princeton, 1982: 814-815.
85. Maier, Michael. *Atalanta Fugiens,* Emblem 21 plus text; transcribed by Hereward Tilton and introduced by Adam McLean. Source: The Alchemy Web Site CD-Rom.
86. *The Consideratio Brevis of Philip à Gabella*. Translated from Latin by Christopher Atton. Published with the first edition of the *Confessio Fraternitati*s at Cassel (1615).
87. Hollandus, Johann Isaac. *A Work of Saturn*. Transcribed by Joshua Ben Arent; published in *Of natural & supernatural things*. London,1670.
88. Paracelsus, Theophastus. *The Aurora of the Philosophers,* Transcribed by Dusan Djordjevic Mileusnic. Published by J. H. Oxon. London, 1659.
89. Kunwrath, Henry. *The naturall Chymicall Symboll or short Confession of Doctor Kunwrath*. Translated from the Dutch, 1597.
90. Stahl, William Harris. *Macrobius: Commentary on the Dream of Scipio*, Columbia University Press, New York, 1952: 191.
91. Giorgi, Francesco. *Harmonia Mundi,* Venice,1525. Trans. Jocelyn Godwin, *THE HARMONY OF THE SPHERES*, Inner Traditions International, Rochester, Vermont,1993:194.
92. Burges, George. *TIMEUS THE LOCRIAN* in *The Works of Plato: A New Literal Version* Vol. VI. George Bell and Sones, London, 1876: 172-173. Full title: *THE TREATISE OF TIMEUS THE LOCRIAN ON THE SOUL OF THE WORLD AND NATURE*.
93. Burges, George. *TIMEUS THE LOCRIAN* in *The Works of Plato: A New Literal Version* Vol. VI. George Bell and Sones, London, 1876: 150-151.
94. Marsilius Ficinus, 'Liber de Arte Chemica', in the *Theatrum Chemicum,* Vol. 2, Geneva, 1702: 172-183.
95. Bernard of Trevisan, an extract from *Le Texte d'Alchymie et le Songe-Verd*, Paris, 1695: 87-92.
96. Green, Robin S. *Spherical Astronomy*, Cambridge University Press, Cambridge 1985:141.
97. Table 15.6. Planets: Mean Elements. *Explanatory Supplement to the Astronomical Almanac*, eds. S. E. Urban & P. K. Seidelmann, University Science Books, Mill Valley. 2000.

* * * * *

INTIMATIONS OF COMMONALITY IN PLANETARY SYSTEMS

John Nicholas Harris, M.A.
spirasolaris.ca